

# AN INTRODUCTION TO CHROMATIC POLYNOMIALS

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Junior Mathematics Seminar, UWA  
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# OUTLINE

## 1 BACKGROUND

- Basics
- Graph Colouring
- 4 Colours Suffice

## 2 POLYNOMIALS

- Chromatic Function
- Chromatic Roots
- Potts Model

## 3 COMPLEX ZEROS

- Absolute Bounds
- Parameterized Bounds

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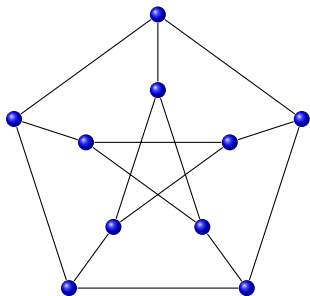
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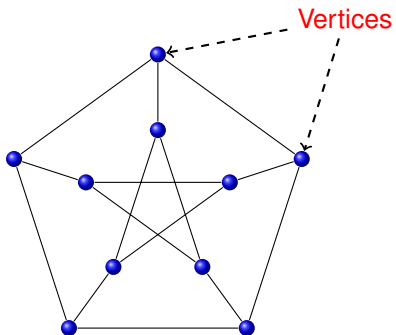
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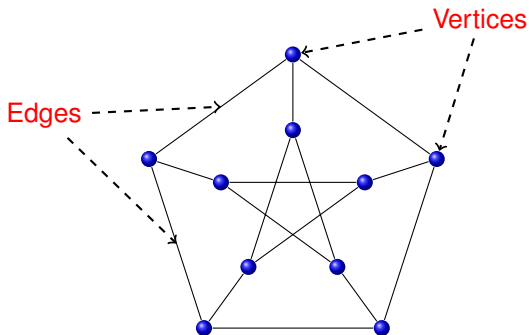
## GRAPHS



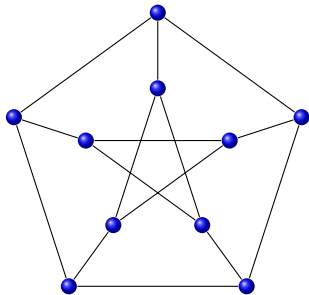
## GRAPHS



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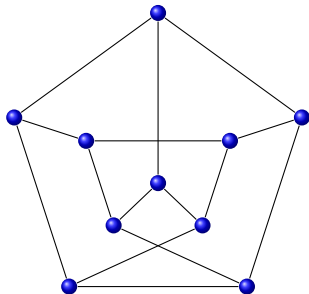
# BASIC TERMINOLOGY



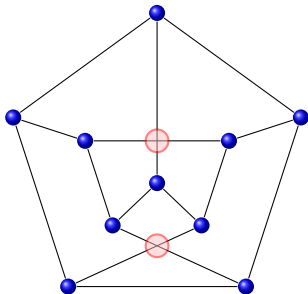
Petersen Graph

- $P$  has 10 *vertices*
- $P$  has 15 *edges*
- Each vertex has 3 *neighbours*
- $P$  is *3-regular* (aka *cubic*)

# POSITION IS IRRELEVANT

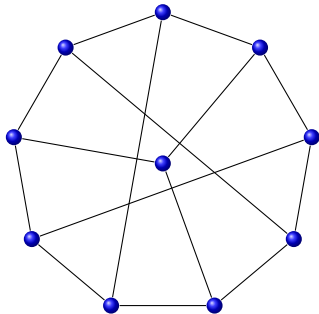
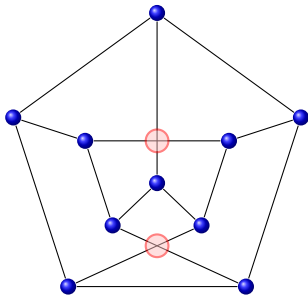


# POSITION IS IRRELEVANT



The Petersen graph is *not planar*

# POSITION IS IRRELEVANT



The Petersen graph is *not planar* and *not hamiltonian*

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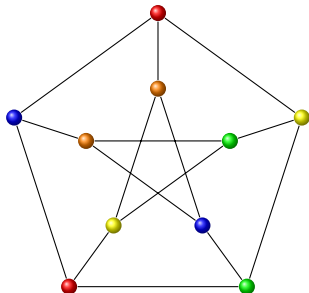
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## COLOURING

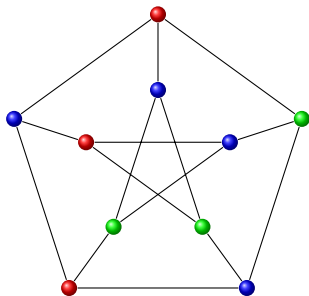
A *proper colouring* of a graph is an assignment of *colours* to the vertices such that *no edge* is monochromatic.



A proper colouring with the five colours  $\{\text{blue, red, green, orange, yellow}\}$ .

# CHROMATIC NUMBER

The *chromatic number*  $\chi(G)$  of a graph is the *minimum number* of colours needed to properly colour  $G$ .



We have exhibited a 3-colouring so  $\chi(P) \leq 3$  and as it obviously cannot be 2-coloured (why?), we get  $\chi(P) = 3$ .

# COLOURING IS HARD

Finding the chromatic number is *hard*: the decision problem

## 3-COLOURING

INSTANCE: A graph  $G$

QUESTION: Does  $G$  have a proper 3-colouring?

is *NP-complete*

It is also hard in *practice* — even graphs with a few hundred vertices are difficult.

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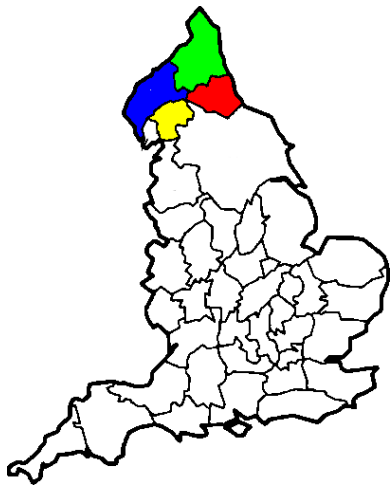
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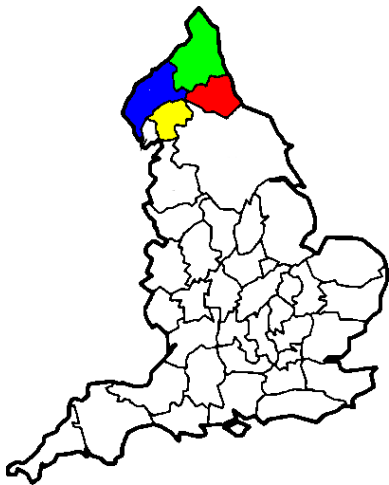
- Absolute Bounds
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# MAP COLOURING



To distinguish regions on a map — in this case a map of the traditional counties of England — the mapmaker *colours* them so that two regions with a common boundary receive different colours.

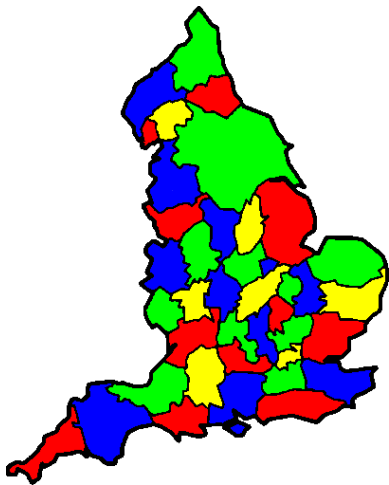
## GUTHRIE'S OBSERVATION



Around 1852, Francis Guthrie noticed that he never needed to use more than 4 colours on any map he tried to colour, and wondered if that was always the case.

This question became known as the *4-colour conjecture*.

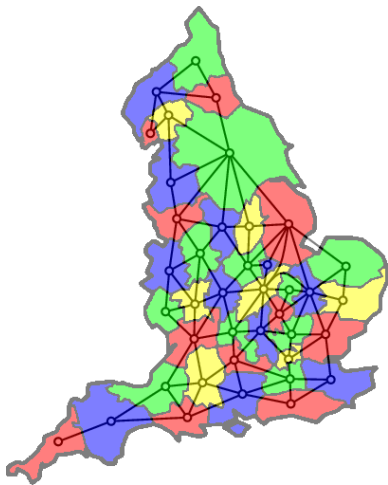
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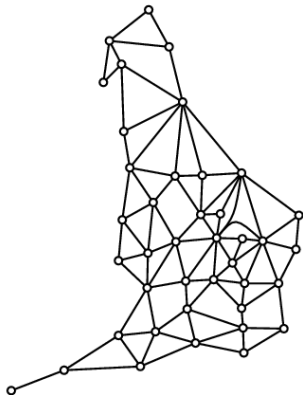
# FROM MAPS TO GRAPHS



The shapes, sizes and positions of the regions are not relevant to the colouring question and so by replacing the map with a *graph* we keep only the essential details.

Graphs arising from maps like this are *planar* — there are no crossing edges.

# FROM MAPS TO GRAPHS



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# THE FOUR-COLOUR THEOREM

The 4-colour conjecture became the most famous problem in graph theory, consumed numerous academic careers and had a far-reaching influence over the development of graph theory.

Finally, more than 120 years later, a heavily computer-aided proof was published:

## THE FOUR COLOUR THEOREM (APPEL & HAKEN 1976)

Every planar graph has a 4-colouring.

# NOT EVERYONE WAS CONVINCED..

Not everyone was convinced  
by the proof. . .

# NOT EVERYONE WAS CONVINCED..

Not everyone was convinced  
by the proof. . .

. . . and some people remain  
that way.

## THE FOUR COLOR CONJECTURE

(It is False)



NELLO CASTELLINI

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## BIRKHOFF

In 1912 Birkhoff introduced the function  $P_G(q)$  such that for a graph  $G$  and positive integer  $q$ ,

$P_G(q)$  is the *number* of proper  $q$ -colourings of  $G$ .



George David Birkhoff  
(1884 – 1944)

# QUANTITATIVE VERSUS QUALITATIVE

This was an attempt to develop *quantitative* tools to *count* the number of colourings of a planar graph, rather than the alternative *qualitative* approach (“Type 1”) of just proving the existence of a 4-colouring.

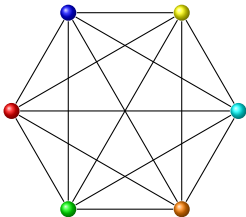
*Type 2.* Here the emphasis is quantitative. Moreover no restriction is made on the number of colors considered. This point of view leads inevitably to certain polynomials each one of which gives the exact number of ways an associated map may be colored in any number of colors. The theory of these so-called “chromatic” polynomials was initiated by Birkhoff in 1912 (Birkhoff

He hoped to be able to find an *analytic* proof that  $P_G(4) > 0$  for any planar graph  $G$ .

# EXAMPLE — COMPLETE GRAPHS

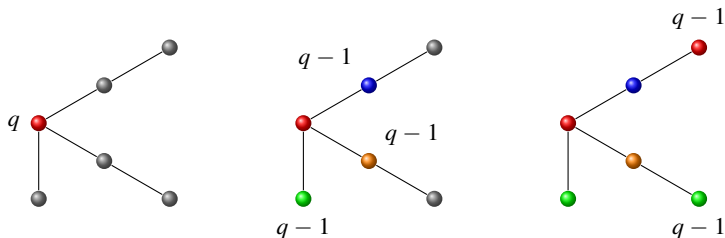
For the complete graph  $K_n$ , where every vertex is joined to all the others, each vertex must be coloured differently, so the total number of  $q$ -colourings is

$$P_{K_n}(q) = q(q-1)(q-2)\dots(q-n+1).$$



## EXAMPLE — TREES

For a *tree* (i.e. a connected graph with no cycles), there are  $q$  choices of colour for an arbitrary first vertex, and then  $q - 1$  choices for each subsequent vertex.

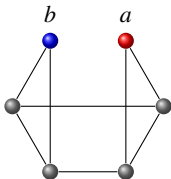


Thus for *any* tree  $T$  on  $n$  vertices, we have

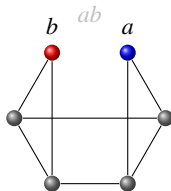
$$P_T(q) = q(q-1)^{n-1}.$$

# ADDITION AND CONTRACTION

Divide the  $q$ -colourings of a graph  $G$  according to whether two (specific) non-adjacent vertices receive the same or different colours.



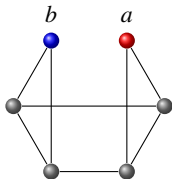
These colourings are in 1-1 correspondence with colourings of  $G + ab$  where vertices  $a$  and  $b$  are joined by an edge.



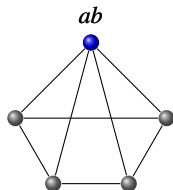
These colourings are in 1-1 correspondence with colourings of  $G/ab$  where  $a$  and  $b$  are coalesced into a single vertex (with edges following).

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# DELETION-CONTRACTION ALGORITHM

Rearranging this shows that for an edge  $ab$

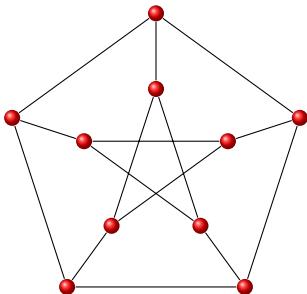
$$P_G(k) = P_{G-ab}(k) - P_{G/ab}(k)$$

Repeated application yields the *deletion-contraction* algorithm, which shows that for an  $n$ -vertex graph  $G$ :

- $P_G(q)$  is a *monic polynomial* of degree  $n$
- $P_G(q)$  has *alternating coefficients*

However this algorithm has *exponential complexity* and therefore is only practical for “small” graphs.

## EXAMPLE — PETERSEN GRAPH



$q^{10} - 15q^9 + 105q^8 - 455q^7 + 1353q^6 - 2861q^5 + 4275q^4 - 4305q^3 + 2606q^2 - 704q$   
 which factors into

$$q(q-1)(q-2)\left(q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352\right)$$

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# REAL ZEROS

Birkhoff & Lewis generalized the Five Colour Theorem:

- Five-Colour Theorem (Heawood 1890)

If  $G$  is *planar* then  $P_G(5) > 0$ .

- Birkhoff-Lewis Theorem (1946)

If  $G$  is *planar* and  $x \geq 5$ , then  $P_G(x) > 0$ .

- Birkhoff-Lewis Conjecture [still unsolved]

If  $G$  is *planar* and  $x \geq 4$  then  $P_G(x) > 0$ .

Leads to studying the *real chromatic zeros* of a graph  $G$  — maybe studying the real numbers  $x$  where  $P_G(x) = 0$  will tell us where  $P_G(x) \neq 0$ ?

# COMPLEX ZEROS

Why not find *all* solutions to the equation

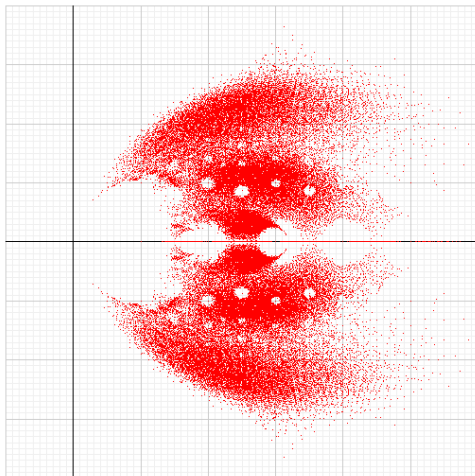
$$P_G(z) = 0?$$

If  $G$  is an  $n$ -vertex graph then  $P_G(z)$  has degree  $n$  and so this equation has  $n$  solutions over the complex numbers — these are the *chromatic zeros* of  $G$ .

<i>Real Parts</i>	<i>Imaginary Parts</i>
0.63596985	0.030695947
0.70105547	0.30191011
0.88769325	0.48067209
0.54666090	1.9521084
0.83394414	1.0924455

First explicit mention of complex chromatic zeros appears to be in a 1965 paper of Hall, Siry & Vanderslice.

# CHROMATIC ZEROS OF 9-VERTEX GRAPHS



# FUNDAMENTAL QUESTIONS

The two fundamental questions prompted by trying to understand the patterns in plots such as this are:

- Are there *absolute bounds* on the root-location independent of the graph structure?
- Can we find bounds on the root-location in terms of *graph parameters*?

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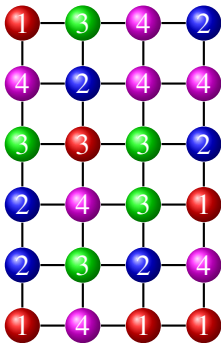
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# POTTS MODEL

The *q-state Potts model* models a physical system as collection of interacting “spins”, each taking on one of  $q$  distinct values, located on a regular lattice grid.



Any possible configuration

$$\sigma : V \rightarrow \{1, 2, \dots, q\}$$

has *Boltzmann weight* given by

$$\prod_{e=xy} (1 + v_e \delta(\sigma(x), \sigma(y)))$$

[i.e. edge  $e$  contributes  $1 + v_e$  if it joins equal spins]

# PARTITION FUNCTION

The *partition function* is the sum of the Boltzmann weights:

$$Z_G(q, \{v_e\}) = \sum_{\sigma: V \rightarrow \{1, 2, \dots, q\}} \prod_{e=xy} (1 + v_e \delta(\sigma(x), \sigma(y)))$$

If we put  $v_e = -1$  for every edge — physically corresponding to the *zero temperature limit of the antiferromagnetic Potts model* — then we get

$$Z_G(q, -1) = P_G(q).$$

This is no mere accident — in one of the amazing examples of “the unreasonable effectiveness of mathematics”, the full 2-variable partition function is equivalent to the 2-variable Tutte polynomial of graphs and matroids.

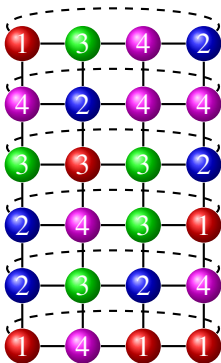
# PHASE TRANSITIONS

A *phase transition* in a physical system occurs when continuous variation in a control parameter yields a discontinuity in its observed behaviour.

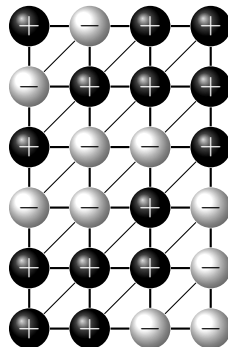
Statistical physicists are interested in complex zeros because a phase transition can only occur at a *real limit point* of the *complex zeros* of the partition function.

Hence a *zero-free region* for a family of graphs provides evidence that phase-transitions cannot occur in that region of parameter space — such theorems are called *Lee-Yang theorems*.

# POTTS MODELS



The lattice may be *periodic* (i.e. wrap around).



The  $q = 2$  case is known as the Ising model.<sup>1</sup>

<sup>1</sup>invented by Ising's supervisor Lenz

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# ABSOLUTE BOUNDS

For many years, it was thought that chromatic zeros were restricted to the right half-plane

$$\operatorname{Re}(z) > 0.$$

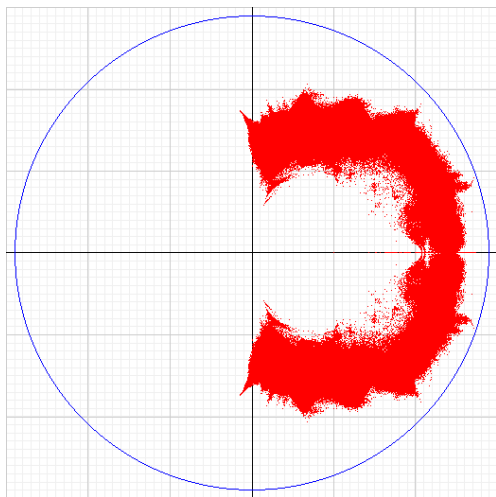
Ron Read<sup>2</sup> and I disproved this in 1988 with *high-girth cubic graphs* as examples.

For several years after that, papers appeared proving the existence of chromatic zeros with increasingly large negative real part.

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<sup>2</sup>For years Ron had a sign on his office door just saying “Please Read”

## CUBIC GRAPHS ON 20 VERTICES



# SOKAL'S SECOND RESULT

This game ground to a halt in spectacular fashion, when the statistical physicist Alan Sokal proved:

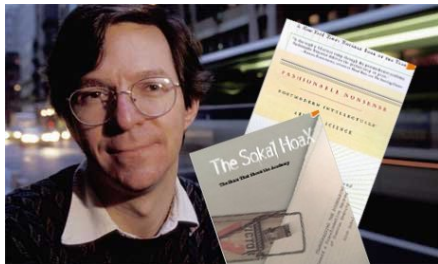
## THEOREM (SOKAL 2000)

Chromatic zeros are dense in the whole complex plane.

The most surprising part of this theorem is that just one very simple class of graphs — generalized  $\Theta$ -graphs — has chromatic roots almost everywhere.

## ALAN SOKAL

Sokal is a brilliant statistical physicist and mathematician, but is most well known for his infamous hoax.

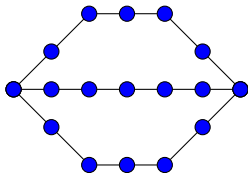


His nonsensical article *Transgressing the Boundaries: Towards a Transformative Hermeneutics of Quantum Gravity* appeared in the eminent postmodernist journal "Social Text" and sparked a furious controversy.

GENERALIZED  $\Theta$ -GRAPHS

Let  $\Theta_{s,p}$  denote the graph obtained by taking two vertices, joining them by  $p$  edges in parallel, and then replacing each edge by  $s$  edges in series.

So  $\Theta_{6,3}$  consists of three 6-edge paths with endpoints identified:



The zeros of generalized  $\Theta$ -graphs are dense everywhere *except possibly* the region  $|z - 1| < 1$ .

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# GENERALIZING BROOKS' THEOREM

A classic early result of graph theory is Brooks' 1941 theorem that if  $G$  has maximum degree  $\Delta(G)$  then

$$\chi(G) \leq \Delta(G) + 1.$$

Biggs, Damerell & Sands (1971) conjectured the existence of a function  $f(r)$  such all zeros of  $P_G(z)$  lie in the region

$$|z| \leq f(r)$$

whenever  $G$  has maximum degree  $r$ .

# SOKAL'S FIRST RESULT

Their conjecture was eventually confirmed:

## THEOREM (SOKAL 1999)

If  $G$  is a graph with maximum degree  $\Delta$  and second largest degree  $\Delta_2$  then all zeros of  $P_G(z)$  lie in the region

$$|z| \leq 7.963907\Delta$$

and the region

$$|z| \leq 7.963907\Delta_2 + 1$$

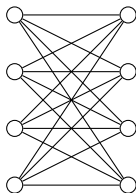
The generalized  $\Theta$ -graphs show that there *cannot* be any bound as a function of  $\Delta_3$ .

# THE ANSWER

I have been convinced for over 20 years that the *real* answer is

## CONJECTURE (ROYLE c.1990)

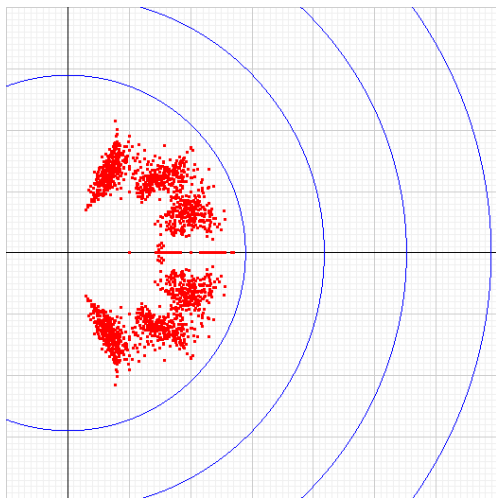
The complete bipartite graph  $K_{r,r}$  contributes the chromatic root of maximum modulus among all graphs of maximum degree  $r$  (excluding  $K_4$  for  $r = 3$ ) — a bound of around  $1.6\Delta$ .

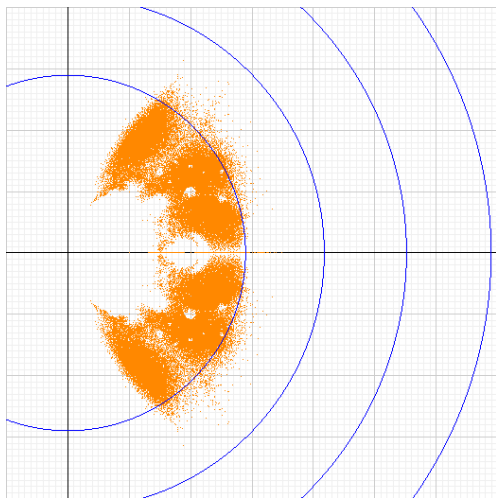


Thus the chromatic root of  $K_{4,4}$  at

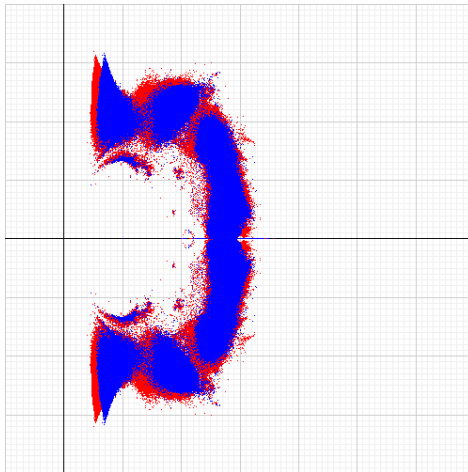
$$z = 2.802489 + 3.097444i$$

should have largest modulus over *all graphs* with  $\Delta = 4$ .

10 VERTEX GRAPHS WITH  $\Delta = 3$ 

10 VERTEX GRAPHS WITH  $\Delta = 4$ 

# QUARTIC GRAPHS ON 15 AND 16 VERTICES



# MANY QUESTIONS REMAIN

I am currently working with Sokal and others to try to improve the bounds on *real* chromatic roots.

The following sequence of (increasingly strong) *conjectures* are all open: if  $G$  has maximum degree  $\Delta$  then

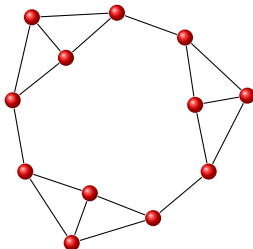
- 1  $P_G(q) > 0$  for all  $q > \Delta$ ;
- 2  $P_G(q)$  *and its derivatives* are positive for  $q > \Delta$ ;
- 3 All roots of  $P_G(q)$ , *real or complex*, have real part at most  $\Delta$ .

Successfully attacking these problems requires *computation*, *graph theory*, some basic *complex analysis*, a talent for *pattern spotting* and a bit of *luck*!

# CUBIC GRAPHS

Which *cubic graph* has the largest real chromatic root?

Computation shows that among the cubic graphs on up to 20 vertices (about 1/2 million of them), this one graph is the "*record holder*" with a real chromatic root at (about) [2.77128607](#).



Can *you* break the record?