On the Derangement graph of $\text{PGL}(2, q)$ acting on the projective line

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November 2009
Joint work with Karen Meagher
The Erdős-Ko-Rado theorem

Let $\Omega = \{x_1, \ldots, x_n\}$ be a finite set of size $n$ and $\mathcal{F}$ a family of $k$-subsets of $\Omega$, for $2k < n$. Suppose further that any two elements of $\mathcal{F}$ intersect in at least one element. Then

(i) $|\mathcal{F}| \leq \binom{n-1}{k-1}$;

(ii) if $|\mathcal{F}| = \binom{n-1}{k-1}$, then all the $k$-sets in $\mathcal{F}$ contain a fixed point $\overline{\omega}$ of $\Omega$.

This result had great impact on combinatorics. There are many generalizations and analogues.

Example

Let $n > k > l > 0$. If for any two elements $A, B$ of $\mathcal{F}$ we have $|A \cap B| \geq l$, then for $n > n_0(k, l)$ we get $|\mathcal{F}| \leq \binom{n-l}{k-l}$ and equality is met if and only if all elements of $\mathcal{F}$ contain an $l$-subset $\overline{x}$ of $X$. 
Example (Hsieh)

Let $\mathcal{F}$ be a family of $k$-dimensional subspaces of $\mathbb{F}_q^n$. Suppose that for all $A, B \in \mathcal{F}$ we have $\dim(A \cap B) \geq l > 0$. Assume $k \leq (n - 1)/2$ or $k < (n - 1)/2$ in case $q = 2$, $l > 1$. Then

$$|\mathcal{F}| \leq \binom{n - l}{k - l}_q.$$

Other analogues of the Erdős-Ko-Rado theorem: for partitions and for permutations.

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Some applications I

(graph theory)
The Kneser graph $K(n, k)$ is the graph whose vertices are the $k$-subsets of a set of size $n$ and two vertices $A, B$ are joined if $A \cap B = \emptyset$. The Petersen graph is an example of a Kneser graph, namely $K(5, 2)$.
The Erdős-Ko-Rado theorem yields that an independent set of maximal size in $K(n, k)$, for $2k < n$, has size $\binom{n-1}{k-1}$. Also, the independent sets of maximal size are fully understood.
Some applications II

(Synch Co-op)

A permutation group is non-separating if and only if there is a graph $X$, neither complete nor null, satisfying $\omega(X)\alpha(X) = |X|$ and $G \leq \text{Aut}(X)$. We recall that $\omega(X)$ is the size of a largest clique in $X$ and $\alpha(X)$ is the size of a largest coclique in $X$.

Take $G = \text{Sym}(n)$ acting on $k$-sets (for $2k \leq n$) and $X = K(n, k)$. In this case $\omega(X) = \lfloor n/k \rfloor$ and $\alpha(X) = \binom{n-1}{k-1}$ (by the EKR theorem). In particular, if $k$ divides $n$, then $G$ is non-separating.
Erdős-Ko-Rado theorem for permutation groups

(proposed first by M. Deza and P. Frankl)

Given two permutations $g, h$ in $\text{Sym}(n)$, we say that $g, h$ are intersecting if $\text{fix}(g^{-1}h) \neq \emptyset$.

What is the maximal size of an intersecting set of permutations in $\text{Sym}(n)$?

Maybe...$(n - 1)!$

What are the sets attaining this bound?

Maybe...the cosets of the stabilizer of a point
Graph-theoretic terminology

Let $D$ be the set of derangements of $\text{Sym}(n)$ (i.e. fixed-point-free elements). The derangement graph $\Gamma_{\text{Sym}(n)}$ of $\text{Sym}(n)$ is the graph whose vertices are the elements of $\text{Sym}(n)$ and whose edges are the pairs $\{g, h\}$ such that $g^{-1}h$ is a derangement.

Note that the right regular representation of $\text{Sym}(n)$ is a subgroup of $\text{Aut}(\Gamma_{\text{Sym}(n)})$. So, $\Gamma_{\text{Sym}(n)}$ is a Cayley graph, i.e. $\Gamma_{\text{Sym}(n)} = \text{Cay}(\text{Sym}(n), D)$.

An independent set for $\Gamma_{\text{Sym}(n)}$ is simply an intersecting set for $\text{Sym}(n)$. 

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Lemma (clique-coclique bound)

Let \( \Gamma \) be vertex-transitive graph, \( C \) a clique of \( \Gamma \) and \( S \) an independent set of \( \Gamma \). Then \( |C||S| \leq |\Gamma| \). Equality is met if and only if \( |C^g \cap S| = 1 \), for every \( g \in \text{Aut}(\Gamma) \).

In the derangement graph \( \Gamma_{\text{Sym}(n)} \), any regular subgroup \( C \) of \( \text{Sym}(n) \) is a clique of size \( n \). Thence, if \( S \) is an independent set, we get

\[
|S| \leq \frac{n!}{n} = (n - 1)!.
\]

Hard to understand whether the independent sets of maximal size are cosets of the stabilizer of a point.
Cameron-Ku and Larose-Malvenuto proved that every independent set of maximal size of $\Gamma_{\text{Sym}(n)}$ is the coset of the stabilizer of a point. (Now we also have a one page proof due to J. Wang and S. J. Zhang)

More recently, Godsil-Meagher proved the same result using the character theory of $\text{Sym}(n)$. 

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The derangement graph of $\operatorname{PGL}(2, q)$ acting on the projective line

The definition of derangement graph can be given on every permutation group $G$ (probably transitive, in order to have derangements!).

In the rest of the talk, in order to describe the method used by Godsil-Meagher, we study the derangement graph of $\operatorname{PGL}(2, q)$ on its action on the projective line.

**Theorem (K.Meagher, P.S.)**

*Every independent set has size at most $q(q – 1)$. The only sets that meet this bound are the cosets of the stabilizer of a point.*
Example

Let $G$ be a permutation group of degree $n$. It is not always the case that $\alpha(\Gamma_G) = |G|/n$.

For instance if $G = \text{Alt}(4)$ acting on 2-sets, then Klein subgroup of $G$ is an independent set of size 4.
Let $\Gamma = \text{Cay}(G, D)$ be a normal Cayley graph, i.e. $D$ is a union of conjugacy classes of $G$.

**Lemma**

The spectrum of $\Gamma$ is

$$\left\{ \frac{1}{\chi(1)} \sum_{g \in D} \chi(g) \mid \chi \in \text{Irr}(G) \right\}.$$  

If $\tau$ is an eigenvalue and $\chi_1, \ldots, \chi_r$ are the characters affording $\tau$, then the multiplicity of $\tau$ is $\sum_{i=1}^r \chi_i(1)^2$.
Lemma (Hoffman bound)

Let $\tau$ be the minimum eigenvalue of $\Gamma$. Assume that $\Gamma$ has valency $d$ and has $v$ vertices. If $S$ is an independent set of $\Gamma$, then

$$|S| \leq \frac{v}{1 - \frac{d}{\tau}}.$$

The equality is met if and only if $v_S - \frac{|S|}{v}v_G$ is in the $\tau$ eigenspace, (here $v_S$ and $v_G$ denote the characteristic vectors of $S$ and $G$).

In particular, if the bound is attained, then looking for independent sets of maximal size is the same as looking for $\{0, 1\}$-vectors in $\langle v_G \rangle + \tau$-eigenspace.
Sometimes Hoffman’s bound is more “useful” than the clique-coclique bound.

For instance, take $G = \text{PSL}(2, q)$ and $\Gamma = \Gamma_G$ the derangement graph of $G$, for $q$ odd. The stabilizer of a point in $\Gamma$ has size $q(q - 1)/2$, therefore $\Gamma$ has independent sets of size $q(q - 1)/2$.

But, if $q \equiv 1 \mod 4$, then $\Gamma$ has no cliques of size $q + 1$. So, one cannot prove using the clique-coclique bound that the size of an independent set of $\Gamma$ is at most $q(q - 1)/2$.

Nevertheless, the minimum eigenvalue of $\Gamma$ is $-(q - 1)^2/4$ and so Hoffman’s bound gives

$$\alpha(\Gamma) \leq \frac{q(q^2 - 1)/2}{1 - \frac{q(q-1)^2/4}{-(q-1)^2/4}} = q(q - 1)/2.$$
The character table of $\text{PGL}(2, q)$ is known (Jordan-Schur). R. Steinberg determined the character table of $\text{PGL}(3, q), \text{PGL}(4, q)$. Finally, the character table of $\text{PGL}(n, q)$ was found by A. Green (not with the $l$-adic cohomology methods of Deligne-Lusztig).

<table>
<thead>
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<th>Name</th>
<th>Nr.</th>
<th>$1$</th>
<th>$u$</th>
<th>$d_x$</th>
<th>$v_r$</th>
</tr>
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<td>$\lambda_1$</td>
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<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>$1$</td>
<td>$q$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\eta\beta$</td>
<td>$\frac{q}{2}$</td>
<td>$q - 1$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$-\beta(r) - \beta(r^{-1})$</td>
</tr>
<tr>
<td>$\nu\gamma$</td>
<td>$\frac{q}{2} - 1$</td>
<td>$q + 1$</td>
<td>$1$</td>
<td>$\gamma(x) + \gamma(x^{-1})$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table: Character table of $\text{PGL}(2, q)$, for $q$ even.
Using the character table of $G_q = \text{PGL}(2, q)$, we obtain that the minimum eigenvalue $\tau$ of $\Gamma_{G_q}$ is $-q(q - 1)/2$.

By Hoffman’s bound, every independent set of $\Gamma_{G_q}$ has size at most $q(q - 1)$ and the bound is attained.

For $\Gamma_{G_q}$, we get (with some work) that $\langle v_{G_q} \rangle + \tau$-eigenspace coincides with the vector space spanned by the characteristic vectors of the cosets of the stabilizer of a point.

Therefore, by Hoffman’s bound, the vector space spanned by the characteristic vectors of the independent sets of size $q(q - 1)$ coincides with the vector space spanned by the characteristic vectors of the cosets of the stabilizer of a point.
Information on the minimum eigenvalue $\tau$ of $\Gamma_G$ can be obtained for many classes of groups. For example, $\tau$ can be easily obtained for the groups $G$ where we character table is explicitly given (as $\text{PSL}(2, q)$, $\text{PGL}(2, q)$, Suzuki groups, Ree groups...). Also, $\tau$ can be obtained for those groups whose character theory is "well-understood" (such as $\text{Sym}(n)$ or $\text{Alt}(n)$).
Idea of the rest of the proof

Let $S$ be an independent set of size $q(q - 1)$. Since $Sg$ is an independent set of size $q(q - 1)$, we may assume that $1 \in S$. We have to solve the following equation:

\[
\begin{pmatrix}
1 & 0 \\
0 & M \\
B & C
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}
= v_S
= \begin{pmatrix}
1 \\
0 \\
t
\end{pmatrix}.
\]
Solving the previous equation is very hard in general.

Assume that whenever $Mw = 0$ we obtain $CW = 0$. Then we simply have to solve

$$Bv = t,$$

where $t$ is a $\{0, 1\}$-vector.

This can be done for many families of groups.

For instance, if for every point $x \in \Omega$ the group $G$ contains a permutation fixing $x$ and acting fixed-point-freely on $\Omega \setminus \{x\}$, then $v$ has at most one non-zero entry. Therefore $S$ is the stabilizer of a point.

In particular, this is the case for Sym, Alt, PSL, PGL, . . .
The hypothesis on $M$ (i.e. $Mw = 0$ implies $Cw = 0$) is very strong. But in the case of $G = \text{Sym}(n), \text{Alt}(n)$, the matrix $M$ does indeed have this property.

**Proof.** It is easy to choose explicitly $n(n - 1)$ derangements of $\text{Sym}(n)$ such that the submatrix $M'$ obtained from $M$ by considering these rows has the same rank as $M$ and where the kernel of $M'$ is easy to compute. Now, check that $M'w = 0$ implies $Cw = 0$.  

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If $G = \text{PGL}(2, q)$, then the kernel of $M$ is contained in the kernel of $C$. To prove this write $N = M^T M$. The matrix $N$ is $G$-invariant and so the irreducible complex characters appearing as a component of the permutation character of $G$ acting on distinct ordered pairs of points give rise to a basis of eigenvectors for $N$. Now, that a basis of eigenvectors is given, it “suffices” to determine the eigenvalue corresponding to each eigenvector/character. In particular, we obtain an explicit description of the kernel of $M$. 

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A remark for finite geometers

If $q$ is even, then $N$ has only 4 eigenvalues and actually $N$ spans a Bose-Mesner algebra of an association scheme of rank 4 constructed using the action of $\text{PGL}(2, q)$ on an ovoid in $\text{PG}(3, q)$.

Is there such a connection when $q$ is odd?
Conjecture

Any independent set of maximal size of the derangement graph of \( \text{PGL}(n + 1, q) \) acting on the projective space \( \mathbb{P}^n_q \) is either the coset of the stabilizer of a point or the coset of the stabilizer of a hyperplane.
If $G$ is the affine general linear group $AGL(1, q)$, then its derangement graph has $q^{q-1}$ independent sets of maximal size (only $q^2$ are cosets of the stabilizer of a point).

An example that seems particularly interesting is the Suzuki group $G = Suz(q)$ acting on the Tits ovoid. Most of the ingredients are already in place in order to prove that every independent set of maximal size is the coset of the stabilizer of a point.

Missing: given $\alpha, \beta, \gamma, \delta$, determine the number of derangements of $G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$.