Graph-restrictive permutation groups and the PSV Conjecture

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A Theorem of Tutte

(1947, 1959)

Let $\Gamma$ be a finite connected cubic graph with an arc-transitive group $G$ of automorphisms. Then $|G_v| \leq 48$.

Corollary: $|G| \leq 48|V\Gamma|$. 
Graph-restrictive

Γ a finite connected graph with $G \leq \text{Aut}(\Gamma)$ transitive on vertices.

$G^\Gamma(\nu)$ is the permutation group induced on $\Gamma(\nu)$ by $G_\nu$.

Given a permutation group $L$, we say that the pair $(\Gamma, G)$ is locally $L$ if $G^\Gamma(\nu) \cong L$ for all vertices $\nu$.

We say that $L$ is graph-restrictive if there is a constant $C$ such that for all locally $L$ pairs $(\Gamma, G)$, we have that $|G_\nu| \leq C$.

Tutte: $C_3$ and $S_3$ are graph-restrictive.
A nonexample

\[ \text{Aut}(\Gamma) = S_2 \text{ wr } D_{2n} \]

\[ \text{Aut}(\Gamma)^{(v)}_\Gamma = D_8 \]

\[ |\text{Aut}(\Gamma)_v| = 2^{n-1}.2 \]
An equivalent definition

$G_v^{[i]}$ is the kernel of the action of $G_v$ on the set of all vertices at distance at most $i$ from $v$.

$L$ is graph-restrictive if and only if there is some constant $k$ such that for all locally $L$ pairs $(\Gamma, G)$ we have $G_v^{[k]} = 1$.

Given an edge $\{v, w\}$, $G_{vw}^{[1]}$ is the kernel of the action of $G_{vw}$ on $\Gamma(v) \cup \Gamma(w)$. 
Some graph-restrictive groups

- Any regular group.
- Gardiner (1973): Any transitive subgroup of $S_4$ other than $D_8$.
- Sami (2006): $D_{2n}$ for $n$ odd.
- Trofimov, Weiss: any 2-transitive group.
- Verret (2009): Groups $L$ such that $L = \langle L_x, L_y \rangle$ and $L_x$ induces $C_p$ on $y^{L_x}$ for some prime $p$ ($p$-subregular).

$D_{2n}$, for $n$ odd, is 2-regular
Primitive groups and generalisations

Let $G \leq \text{Sym}(\Omega)$.

- Call $G$ primitive if the only partitions of $\Omega$ that it preserves are the trivial ones $\{\Omega\}$ and $\{\{\omega\} \mid \omega \in \Omega\}$.
- Call $G$ quasiprimitive if every nontrivial normal subgroup is transitive.
- Call $G$ semiprimitive if every nontrivial normal subgroup is transitive or semiregular.
Semiprimitive groups

Initially studied by Bereczky and Maróti.

Examples include:

- primitive and quasiprimitive groups;
- regular groups;
- Frobenius groups (that is, all nontrivial elements fix at most one point);
- $\text{GL}(n, p)$ acting on the set of nonzero vectors of $\mathbb{Z}_p^n$. 
Weiss Conjecture

**Weiss Conjecture (1978):** Every primitive group is graph-restrictive.

**Weiss (1979):** If $L$ is a primitive permutation group of affine type on $p^d$ points for $p \geq 5$, then $L$ is graph-restrictive.

**Praeger, Spiga, Verret (2012):** Reduced to a problem about simple groups.

**Praeger, Pyber, Spiga, Szabó (2012):** Weiss conjecture is true if composition factors in $G$ have bounded rank.
What is the correct setting?

**Praeger Conjecture:** Every quasiprimitive group is graph-restrictive.

**Potočnik, Spiga, Verret (2012):** If a transitive group is graph restrictive then it is semiprimitive.

**PSV conjecture:** A transitive group is graph-restrictive if and only if it is semiprimitive.

\( D_8 \) is not semiprimitive as it contains a normal intransitive subgroup isomorphic to \( C_2^2 \).

**Spiga, Verret (2014):** An intransitive group is graph-restrictive if and only if it is semiregular.
Variation on Thompson-Wielandt

Spiga (2012): If $(\Gamma, G)$ is a locally semiprimitive pair and \{v, w\} is an edge such that $G_{vw}^{[1]} \neq 1$ then $G_{vw}^{[1]}$ is a $p$-group.
Let $L$ be a semiprimitive group with a regular normal nilpotent subgroup $K$.

(A group is nilpotent if and only if it is the direct product of its Sylow subgroups.)

**Theorem** Every transitive normal subgroup contains $K$, and every semiregular normal subgroup is contained in $K$.

**Theorem** Let $(\Gamma, G)$ be a locally $L$ pair with $|K|$ coprime to 6. Then $G^{[1]}_{vw} = 1$ and so $L$ is graph-restrictive.
Semiprimitive groups of this type include:

- affine primitive groups on $p^n$ points for $p \geq 5$;
- Frobenius groups of degree coprime to 6;
- $P \rtimes C_2$ with $P$ a regular abelian $p$-group for $p \geq 5$ and $C_2$ acting by inversion;
- $p^{1+2m} \rtimes \text{Sp}(2m, q)$ with $p \geq 5$.
- $V = \text{GF}(q)^n$ and $G = (V \oplus V \oplus \cdots \oplus V) \rtimes \text{GL}(V)$
Also give detailed information about what a counterexample with order not coprime to 6 must look like.

**Theorem** Let \((\Gamma, G)\) be a locally \(L\) pair where \(L\) is semiprimitive with a regular normal nilpotent subgroup \(K\) and suppose that \(G_{vw}^{[1]} \neq 1\). Then \(L\) contains normal subgroups \(F\) and \(J\) such that \(F < K < J\) and either

- \(G_{xy}^{[1]}\) is a 2-group and \(J/F \cong S_3 \times \cdots \times S_3\), or
- \(G_{xy}^{[1]}\) is a 3-group and \(J/F \cong A_4 \times \cdots \times A_4\).
Potočnik, Spiga and Verret looked at all transitive groups of degree at most 13. The only ones whose status at the time were unknown were:

- $S_3 \wr S_2$ on 9 points (primitive)
- $3^2 \wr 2$ on 9 points (imprimitive)
- $\text{Sym}(5)$ on 10 points (primitive)
- $\text{Sym}(4)$ on 12 points (imprimitive)
A class of Frobenius groups

Let $L$ be the Frobenius group $C_3^n \rtimes C_2$ acting on $3^n$ points with $n \geq 1$.

**Theorem** If $(\Gamma, G)$ is a locally $L$ pair then $G_v^{[4]} = 1$ and so $L$ is graph restrictive.

Tutte’s Theorem is the case $n = 1$. 