Bounding the number of automorphisms of a graph

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Automorphisms of graphs

Γ a locally finite, simple, connected graph.
Vertex set $V\Gamma$, edge set $E\Gamma$, arc set $A\Gamma$
Automorphisms of graphs

$\Gamma$ a locally finite, simple, connected graph.

Vertex set $V\Gamma$, edge set $E\Gamma$, arc set $A\Gamma$

An automorphism of $\Gamma$ is a permutation of the vertices that maps edges to edges.

$\text{Aut}(\Gamma)$ is the group of all automorphisms of $\Gamma$. 
Automorphisms of the Petersen graph

Rotations and reflections gives $D_{10}$. Interchange inside with outside. This gives 20 automorphisms.
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Interchange inside with outside

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$\text{Aut}(\Gamma) = S_5$
Symmetry conditions

Given $G \leq \text{Aut}(\Gamma)$ then $G$ is

vertex-transitive: transitive on $V\Gamma$
edge-transitive: transitive on $E\Gamma$
arc-transitive: transitive on $A\Gamma$
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Edge-transitive but not vertex-transitive implies that $\Gamma$ is bipartite and $G$ has two orbits on vertices.
s-arc transitive graphs

An s-arc in a graph is an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that \(v_i \sim v_{i+1}\) and \(v_{i-1} \neq v_{i+1}\).
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\[\text{K}_4\] is 2-arc transitive but not 3-arc transitive. The Petersen graph is 3-arc transitive but not 4-arc transitive.
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An \textbf{s-arc} in a graph is an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that \(v_i \sim v_{i+1}\) and \(v_{i-1} \neq v_{i+1}\).

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How many automorphisms?

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The vertex stabiliser \( G_\nu \) is the set of all elements of \( G \) that fix \( \nu \).
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Note that if \( \Gamma \) has valency \( d \) and \( G \) is arc-transitive then also have \( |G_v| = d|G_{vw}| \).
A Theorem of Tutte
(1947,1959)

**Theorem** Let $\Gamma$ be a connected cubic graph with an arc-transitive group $G$ of automorphisms such that $G_v$ is finite. Then $|G_v| = 3.2^s$ for some $s \leq 4$. 
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**Corollary** A finite cubic graph is at most 5-arc transitive.
Djoković and Miller (1980): Determined the possible structures of finite vertex and edge-stabilisers for cubic arc-transitive graphs:

- Only 7 possibilities for the pair \((G_v, G_e)\) with \(e = \{u, v\}\).
- In particular, \(G\) is a quotient of one of 7 finitely presented groups.
Possibilities for \((G_v, G_e)\)

<table>
<thead>
<tr>
<th>(s)</th>
<th>(G_v)</th>
<th>(G_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(C_3)</td>
<td>(C_2)</td>
</tr>
<tr>
<td>2</td>
<td>(S_3)</td>
<td>(C_2 \times C_2)  or (C_4)</td>
</tr>
<tr>
<td>3</td>
<td>(S_3 \times C_2)</td>
<td>(D_8)</td>
</tr>
<tr>
<td>4</td>
<td>(S_4)</td>
<td>(D_{16})  or (QD_{16})</td>
</tr>
<tr>
<td>5</td>
<td>(S_4 \times C_2)</td>
<td>((D_8 \times C_2) \rtimes C_2)</td>
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</tbody>
</table>
**Applications**

**Conder and Dobcsányi (2002):** Determined all cubic arc-transitive graphs on at most 768 vertices:

- $|\text{Aut}(\Gamma)| \leq 768.48 = 36864$
- So need to find all normal subgroups of index at most 36864.
Applications

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- $|\text{Aut}(\Gamma)| \leq 768.48 = 36864$
- So need to find all normal subgroups of index at most 36864.
- Conder has subsequently enumerated all such graphs of order at most 2048.
Goldschmidt (1980): Determined the possible structures of finite pairs \((G_u, G_v)\) for adjacent vertices \(u, v\) in cubic edge-transitive graphs:

- only fifteen possibilities
- \(|G_v| \leq 384\)
Local actions

$\Gamma(v)$ is the set of neighbours of $v$.

\[ G_v^{\Gamma(v)} \text{ is the permutation group induced on } \Gamma(v) \text{ by } G_v, \text{ called the local action of } G_v. \]
Local actions

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$G_v^{\Gamma(v)}$ is the permutation group induced on $\Gamma(v)$ by $G_v$, called the local action of $G_v$.

If $G$ is vertex-transitive then all the $G_v^{\Gamma(v)}$ are isomorphic.
Local actions

Γ connected, $G \leq Aut(\Gamma)$ vertex-transitive

- Given a permutation group $L$, we say that the pair $(\Gamma, G)$ is locally $L$ if $G^{\Gamma(v)} \cong L$ for all vertices $v$.
- Given some permutation group property $\mathcal{P}$, we say that $(\Gamma, G)$ is locally $\mathcal{P}$ if $G^{\Gamma(v)}$ has property $\mathcal{P}$ for all vertices $v$. 

Lemma: $\Gamma$ is 2-arc transitive if and only if $(\Gamma, \text{Aut}(\Gamma))$ is locally 2-transitive.
Local actions

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- Given some permutation group property \(\mathcal{P}\), we say that \((\Gamma, G)\) is locally \(\mathcal{P}\) if \(G^{\Gamma(v)}\) has property \(\mathcal{P}\) for all vertices \(v\).

**Lemma** \(\Gamma\) is 2-arc transitive if and only if \((\Gamma, \text{Aut}(\Gamma))\) is locally 2-transitive.
Weiss Conjecture

Let $G \leq \text{Sym}(\Omega)$.

Call $G$ primitive if the only partitions of $\Omega$ that it preserves are the trivial ones $\{\Omega\}$ and $\{\{\omega\} \mid \omega \in \Omega\}$. 
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**Weiss Conjecture (1978):** There is some function $f(d)$ such that for every locally primitive pair $(\Gamma, G)$ of valency $d$ and $G_v$ finite we have $|G_v| \leq f(d)$. 
Verret: We say that $L$ is graph-restrictive if there is a constant $C$ such that for all locally $L$ pairs $(\Gamma, G)$ with $G_v$ finite, we have that $|G_v| \leq C$. 
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Verret: We say that $L$ is **graph-restrictive** if there is a constant $C$ such that for all locally $L$ pairs $(\Gamma, G)$ with $G_v$ finite, we have that $|G_v| \leq C$.

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- The Weiss Conjecture asserts that every primitive group is graph-restrictive.
A nonexample

Wreath graphs

\[ \text{Aut}(\Gamma) = S_2 \text{ wr } D_{2n} \]
\[ \text{Aut}(\Gamma)^{\Gamma(v)} = D_8 \]
\[ |\text{Aut}(\Gamma)_v| = 2^{n-1} \cdot 2 \]
An equivalent definition

$G_{v}^{[i]}$ is the kernel of the action of $G_{v}$ on the set of all vertices at distance at most $i$ from $v$.

If $\Gamma$ is connected and $G_{v}^{[i]} = G_{v}^{[i+1]}$ for some $i$, then $G_{v}^{[i]} = 1$. 

Lemma

$L$ is graph-restrictive if and only if there is some constant $k$ such that for all locally $L$ pairs $(\Gamma, G)$ with $G_{v}$ finite, we have $G_{v}^{[k]} = 1$.

Tutte: For cubic arc-transitive graphs with $G_{v}$ finite we have $G_{v}^{[3]} = 1$. 
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Some graph-restrictive groups

- Any regular group.
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- Gardiner (1973): Any transitive subgroup of $S_4$ other than $D_8$. 
- Sami (2006): $D_{2n}$ for $n$ odd.
- Verret (2009): Groups $L$ such that $L = \langle L_x, L_y \rangle$ and $L_x$ induces $C_p$ on $y$, $L_x$ for some prime $p$ ($p$-subregular).
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Progress on the Weiss Conjecture

Weiss (1979): If $L$ is a primitive permutation group of affine type on $p^d$ points for \( p \geq 5 \), then $L$ is graph-restrictive.

Trofimov, Weiss: Any 2-transitive group is graph-restrictive \((G[v] = 1)\).

Trofimov, Weiss (1995): $\text{PSL}_n(q)$ acting on $m$-spaces is graph-restrictive.

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What is the correct setting?

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- Call $G$ quasiprimitive if every nontrivial normal subgroup is transitive.
- Call $G$ semiprimitive if every nontrivial normal subgroup is transitive or semiregular.
  (A permutation group $H$ is semiregular on $\Omega$ if $H_\alpha = 1$ for all $\alpha \in \Omega$.)
Semiprimitive groups

Initially studied by Bereczky and Maróti (motivated by an application from universal algebra and collapsing monoids).

Examples include:

• primitive and quasiprimitive groups;
• regular groups;
• Frobenius groups (that is, all nontrivial elements fix at most one point);
• $\text{GL}(n, p)$ acting on the set of nonzero vectors of $\mathbb{Z}_p^n$. 
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**PSV conjecture:** A transitive group is graph-restrictive if and only if it is semiprimitive.
Intransitive local actions

Spiga, Verret (2014): An intransitive group is graph-restrictive if and only if it is semiregular.
The edge-transitive case

$\Gamma$ edge-transitive but not vertex transitive. Edge $\{v, w\}$

Say $(\Gamma, G)$ is locally $[L_1, L_2]$ if $G_{v}^{\Gamma(v)} \cong L_1$ or $L_2$ for all vertices $v$. 

Goldschmidt-Sims Conjecture: If $L_1$ and $L_2$ are primitive then there is a constant $C$ such that if $(\Gamma, G)$ is locally $[L_1, L_2]$ with finite vertex stabilisers then $|G_{vw}| \leq C$.

Morgan, Spiga, Verret (2015): If either $L_1$ or $L_2$ is not semiprimitive then there is no bound on $|G_{vw}|$ for a locally $[L_1, L_2]$ pair $(\Gamma, G)$ with finite stabilisers.
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Given an edge \( \{v, w\} \), \( G_{vw}^{[1]} \) is the kernel of the action of \( G_{vw} \) on \( \Gamma(v) \cup \Gamma(w) \).

**Thompson-Wielandt Theorem:** If \( (\Gamma, G) \) is a locally primitive pair with \( G_v \) finite and \( \{v, w\} \) is an edge, then \( G_{vw}^{[1]} \) is a \( p \)-group for some prime \( p \).
Variation on Thompson-Wielandt

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- **van Bon (2003):** Still holds if \( (\Gamma, G) \) is locally quasiprimitive.
- **Spiga (2012):** Still holds if \( (\Gamma, G) \) is locally semiprimitive.
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Regular nilpotent normal subgroups
joint work with Luke Morgan

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**Theorem** Let $(\Gamma, G)$ be a locally $L$ pair with $G_v$ finite and valency coprime to 6. Then $G^{[1]}_{vw} = 1$ and so $L$ is graph-restrictive.
Semiprimitive groups of this type include:

- affine primitive groups on $p^n$ points for $p \geq 5$ (Weiss);
- Frobenius groups of degree coprime to 6;
- $P \rtimes C_2$ with $P$ a regular abelian $p$-group for $p \geq 5$ and $C_2$ acting by inversion;
- $p^{1+2m} \rtimes \text{Sp}(2m, p)$ with $p \geq 5$;
- $V = \text{GF}(p)^n$ and $G = (V \oplus V \oplus \cdots \oplus V) \rtimes \text{GL}(V)$ with $p \geq 5$. 
More detailed information

joint work with Luke Morgan

Also give detailed information about what a counterexample with valency not coprime to 6 must look like.
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**Theorem** Let $(\Gamma, G)$ be a locally $L$ pair where $L$ is semiprimitive with a regular normal nilpotent subgroup $K$ and suppose that $G_v$ is finite and $G_v^{[1]} \neq 1$. Then $L$ contains normal subgroups $F$ and $J$ such that $F < K < J$ and either

- $G_v^{[1]}$ is a 2-group and $J/F \cong S_3 \times \cdots \times S_3$, or
- $G_v^{[1]}$ is a 3-group and $J/F \cong A_4 \times \cdots \times A_4$. 
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Structure is similar to the cases left open by Weiss for affine primitive groups.
Some ingredients of the proof

Assume $G_{vw}^{[1]} \neq 1$ and aim for a contradiction by trying to find an edge-transitive subgroup with a normal subgroup contained in $G_{vw}$.

- Glauberman’s Theorem on groups that are not Thompson factorisable.
- Stellmacher’s Theorem on $S_4$-free groups
Small groups

Potočnik, Spiga and Verret looked at all transitive groups of degree at most 13. The only ones whose status at the time were unknown were:

- $S_3 \wr S_2$ on 9 points (primitive)
- $3^2 \rtimes 2$ on 9 points (imprimitive)
- $\text{Sym}(5)$ on 10 points (primitive)
- $\text{Sym}(4)$ on 12 points (imprimitive)
Let $L$ be the Frobenius group $C_3^n \rtimes C_2$ acting on $3^n$ points with $n \geq 1$.

**Theorem** If $(\Gamma, G)$ is a locally $L$ pair with $G_v$ finite, then $G_v^{[4]} = 1$ and so $L$ is graph restrictive.
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Since $S_3 = C_3 \rtimes C_2$, Tutte’s Theorem is the case $n = 1$. 