2-Neighbour Transitive Codes

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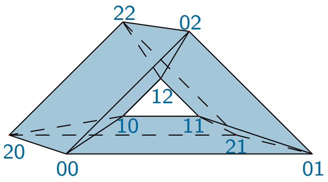
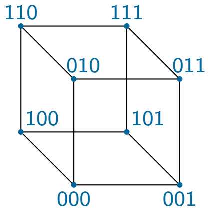
Hamming Graphs

Hamming Graph - $\Gamma = H(m, q)$

- Alphabet - set Q ($|Q| = q$)
- Entries - set M ($|M| = m$)
- Vertex set - strings of length m
- Edge set - pairs of strings which differ in one entry
- Distance - $d(\alpha, \beta)$ length of shortest path from α to β





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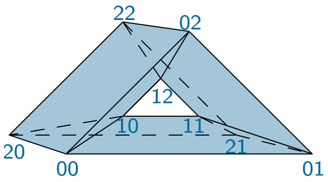
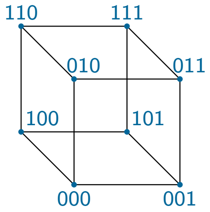
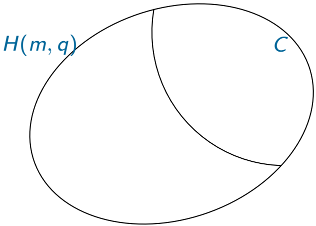




$$\mathrm{Aut}(\Gamma) = \mathrm{Sym}(Q)^m \rtimes \mathrm{Sym}(M)$$

Codes in Hamming Graphs

A *code* C is a subset of the vertices of a graph



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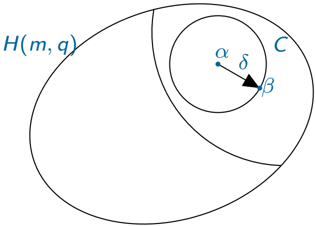

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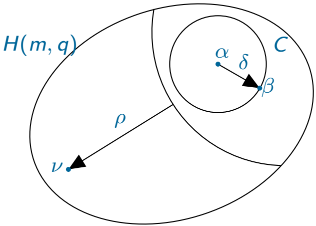
- *covering radius* $\rho = \max\{d(\nu, C) \mid \nu \in H(m, q)\}$



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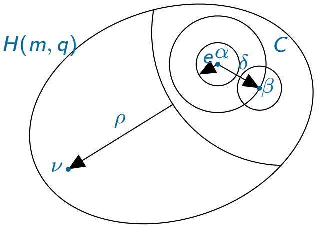

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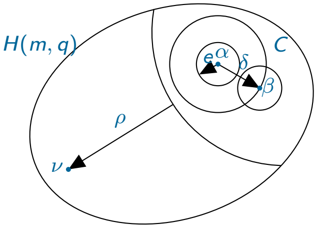
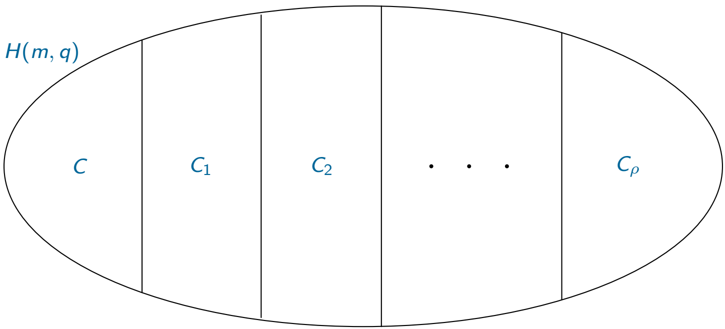


$$\mathrm{Aut}(C) = \mathrm{Aut}(\Gamma)_C$$

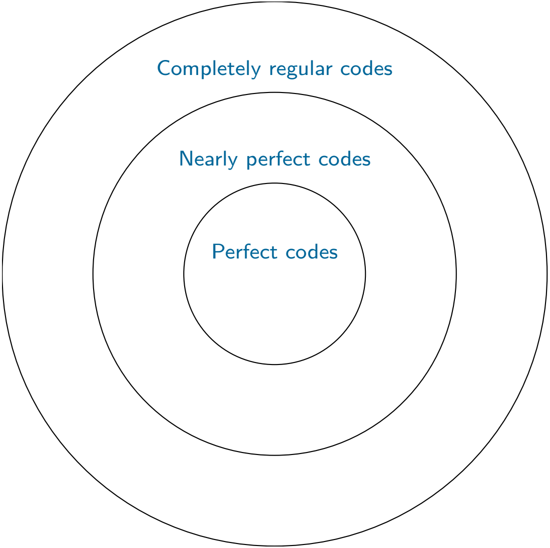
Distance Partition

Define the *set of s-neighbours* C_s to be all vertices $\alpha \in H(m, q)$ with $d(\alpha, C) = s$

The *distance partition*:



Combinatorial Symmetry of Codes



Delsarte (1973)

Goethals (1972)

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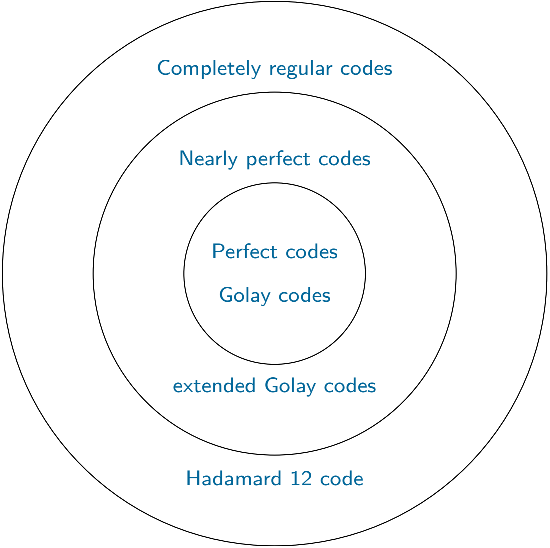
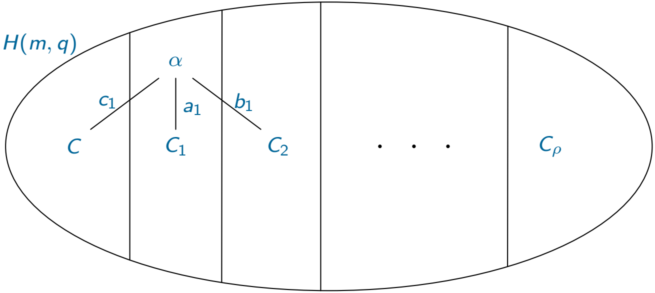


Delsarte (1973)

Goethals (1972)

Completely Regular Codes

Completely regular - for any $i \in \{0, 1, \ldots, \rho\}$, any two vertices in C_i are adjacent to the same number of vertices from each of C_{i-1}, C_i and C_{i+1}



C is *s-regular* if this holds for each $i \in \{0, 1, \ldots, s\}$, for some $s \leqslant \rho$

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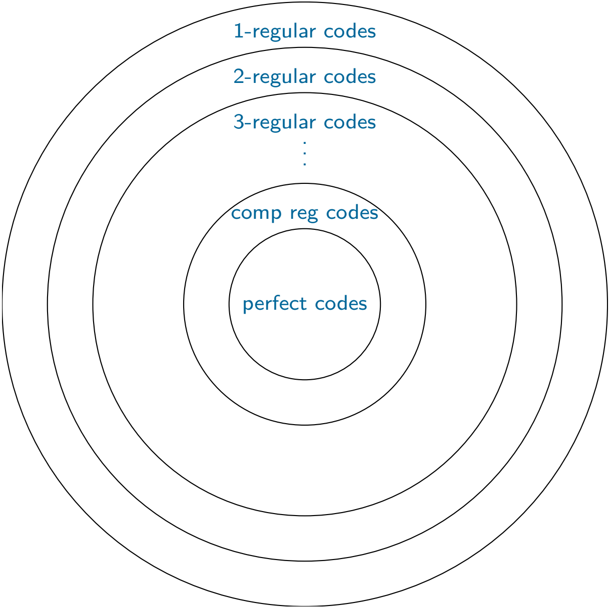
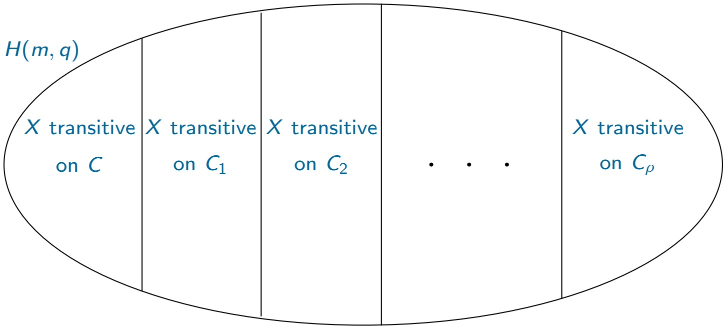


Algebraic Symmetry of Codes

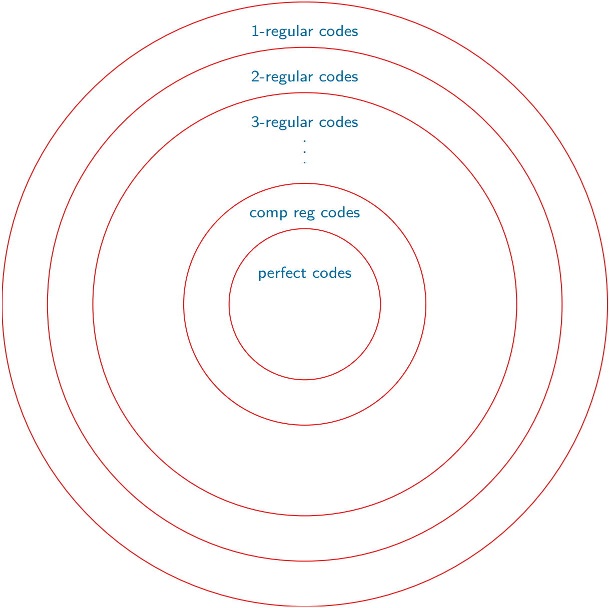
Solé (1987), refined by Giudici and Praeger (2000)

$\exists\ X \leqslant \mathrm{Aut}(C)$ s.t. C_i is an X-orbit for $i \in \{0, 1, \ldots, \rho\}$



C is *s-neighbour transitive* if this instead holds for each $i \in \{0, 1, \ldots, s\}$, for some $s \leqslant \rho$ - Gillespie and Praeger (2011)

Algebraic Symmetry of Codes



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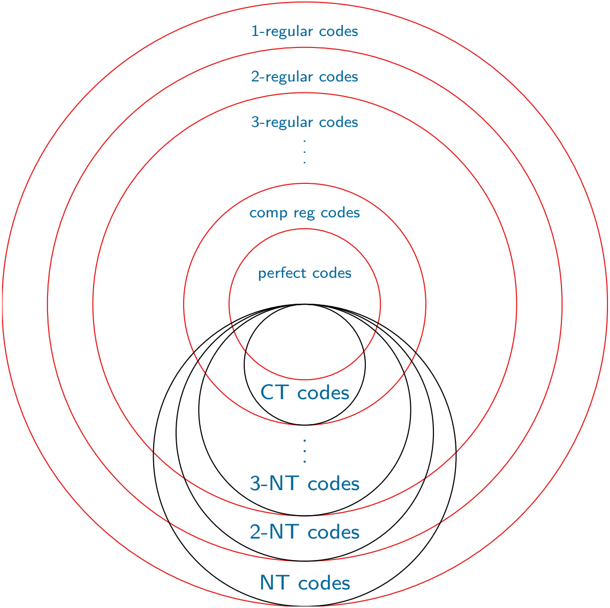

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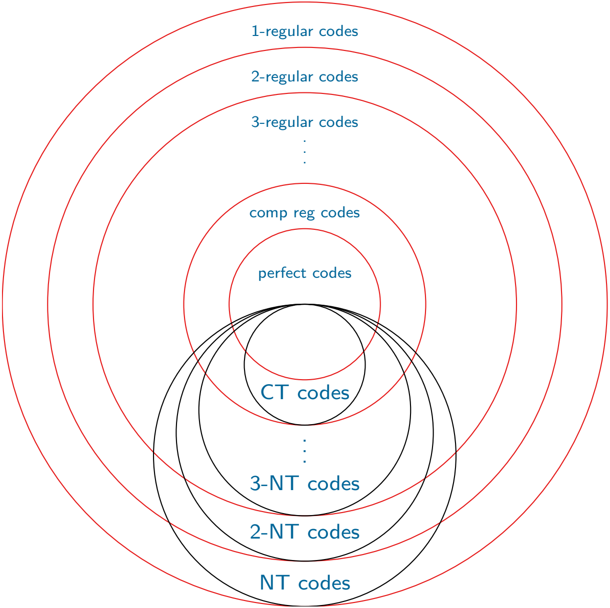
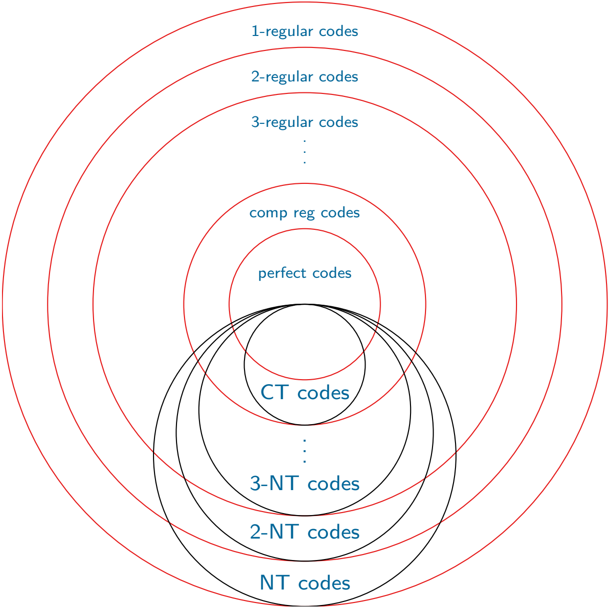


Examples

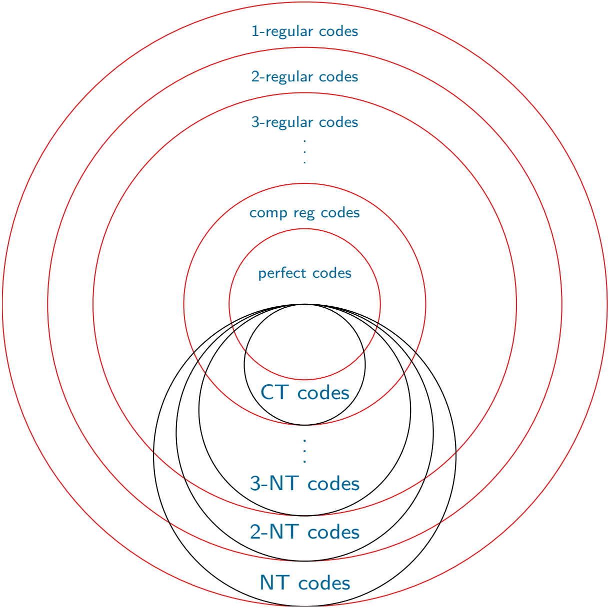
Algebraic Symmetry of Codes



Examples

Preparata codes

Algebraic Symmetry of Codes



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Preparata codes

Hamming-like codes

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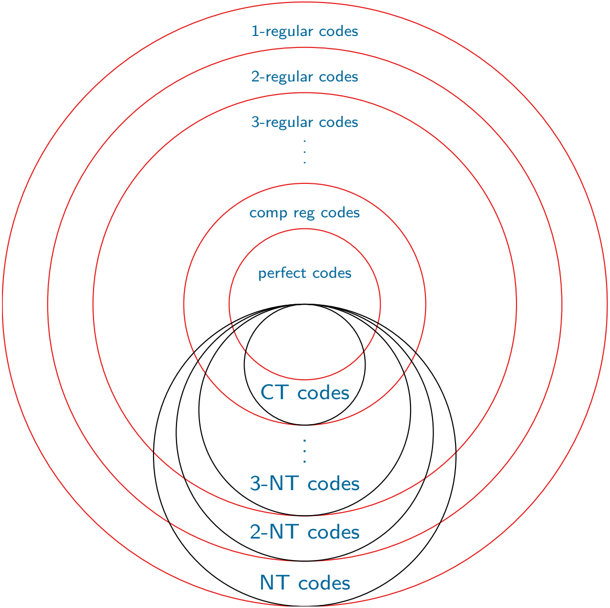

Examples

Preparata codes

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extended Golay codes

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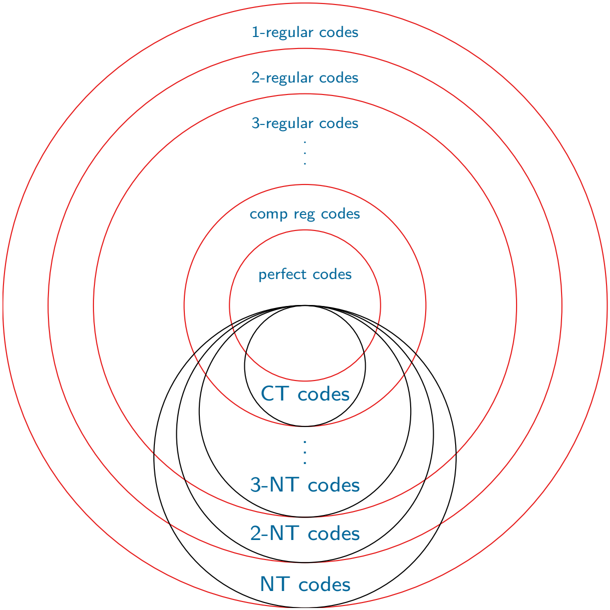

Examples

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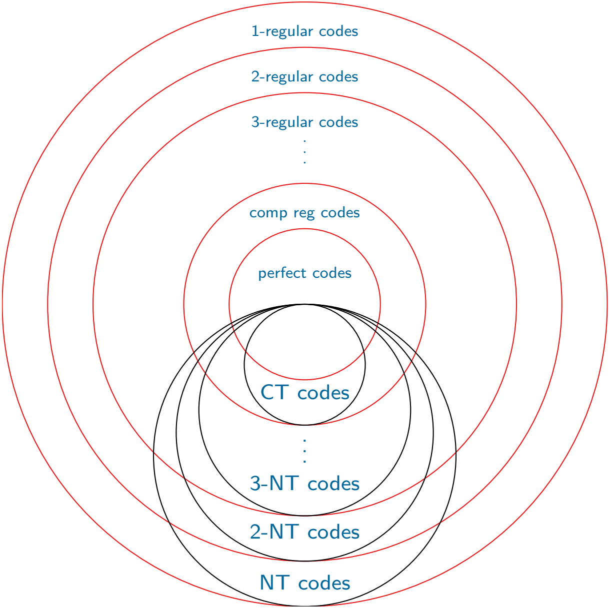

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projective Reed-Muller codes

permutation codes

Reed-Muller codes

Reed-Muller codes:

- vertices - $\{f : \mathbb{F}_q^d \to \mathbb{F}_q\}$
- $M = \mathbb{F}_q^d$
- $Q = \mathbb{F}_q$
- $\mathrm{GL}_d(q)$ acts as 'pure permutations'
- $f^g(x) = f(x^{g^{-1}})$
- $\mathrm{GL}_d(q)$ is 2-transitive on entries
- codes are subspaces defined by the degree of the poly. representing the vertices
- only *guarenteed* to be neighbour transitive

Projective Reed-Muller codes:

- vertices - $\{f : \mathbb{F}_q^d \to \mathbb{F}_q \mid f(ax) = af(x)\}$
- $M =$ 1-dim subspaces of $\mathbb{F}_q^d$
- $Q =$ set of linear fns $\mathbb{F}_q \to \mathbb{F}_q$
- $\mathrm{GL}_d(q)$ acts, but *not* as 'pure permutations'
- $f^g(x) = f(x^{g^{-1}})$
- $\mathrm{GL}_d(q)$ is 2-transitive on weight two vectors
- codes are subspaces defined by the degree of the poly. representing the vertices
- at least 2-neighbour transitive

Suzuki, Ree and Unitary Groups

Codes similar to proj. Reed-Muller with $X_{\mathbf{0}}^M \cong \mathrm{Sz}(q)$, $\mathrm{Ree}(q)$ or $\mathrm{PSU}_3(q)$:

- vertices - $\{f : \mathbb{F}_q^d \to \mathbb{F}_q \mid f(ax) = af(x)\}$
- $M =$ some set of 1-dim subspaces of $\mathbb{F}_q^d$ (inversive plane, Ree unital, isotropic subspaces)
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- codes defined similarly to PRM
- at least 2-neighbour transitive
- new codes (afawk)

Action on Entries

Consider the action X^M of X on the set M of entries:

Lemma

Suppose C is 2-neighbour transitive with minimum distacnce $\delta \geq 5$. Then X^M is 2-homogeneous.

Sketch:

- $X_{\mathbf{0}}$ is transitive on weight two vertices (uses $\delta \geq 5$)
- each weight two codeword is non-zero in two entries
- so $X_{\mathbf{0}}^M$ is 2-homogeneous

Action on the Alphabet Q

Consider the subgroup X_1 stabilising $1 \in M$ and its action X_1^Q on Q:

Lemma

Suppose C is 2-neighbour transitive with minimum distacnce $\delta \geq 5$. Then X_1^Q is 2-transitive.

Sketch:

- Weight δ codewords form a (q-ary) 2-design,
- $\exists$ codeword that starts with $a \in Q^*$,
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- $(X_{\mathbf{0},1})^Q$ transitive on neighbours $\Rightarrow X_1^Q$ 2-transitive.

A 2-Neighbour Transitive Code

The punctured Hadamard 12 code $\mathcal{P}$ is CT ($\rho = 3$).
The even weight subcode of $\mathcal{P}$ is 2-NT ($\rho = 5$).

0	1	2	3	4	5	6	7	8	9	10
1	1	0	1	1	1	0	0	0	1	0

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0	1	0	1	1	0	1	1	1	0	0
0	0	1	0	1	1	0	1	1	1	0
0	0	0	1	0	1	1	0	1	1	1
1	0	0	0	1	0	1	1	0	1	1
1	1	0	0	0	1	0	1	1	0	1
1	1	1	0	0	0	1	0	1	1	0
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0	1	1	1	0	0	0	1	0	1	1
1	0	1	1	1	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0

2-neighbour transitive, but not 3-neighbour transitive.

Entry-Faithful (EF) Codes

Entry faithful - kernel of X^M is trivial

C entry-faithful completely transitive with $\delta \geq 5$ implies C is the repetition code (Gillespie, Giudici and Praeger - 2012)

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Theorem (Gillespie, DH, Giudici, Praeger)

Let C be an X entry faithful 2-neighbour transitive code with $|C| \geq 2$ and $\delta \geq 5$. Then,

- *C is the binary repetition code and $X \cong S_m, M_{22} \rtimes 2$, $X \leqslant \mathrm{A\Gamma L}_d(r)$, or $X \trianglerighteq \mathrm{PSL}_d(r)$; or*
- *C is the even weight subcode of the punctured Hadamard 12 code and $X \cong M_{11}$.*

Alphabet-Almost-Simple Codes

Here the 2-transitive action X_1^Q is almost-simple

Gillespie and Praeger (2012) characterised alphabet-almost-simple (AAS) 1-NT codes with $\delta \geq 3$ by the blocks of imprimitivity of C

Several infinite families of possible blocks are obtained from *permutation codes* - codewords defined by permutations: $\alpha(g) = (1^g, 2^g, \ldots, q^g)$

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Theorem (Gillespie, DH)

Let C be an AAS 2-NT code in $H(m, q)$, with $\delta \geq 3$. Then $m = \delta = 3$, $q \geq 5$ and C is equivalent to the repetition code with these parameters.

Alphabet-Affine Codes

Here the 2-transitive action X_1^Q is affine.

First we considered the case where the kernel of the action X^M is a one-dim. vector space

Theorem (Gillespie, DH, Praeger)

Let C be an alphabet-affine 2-NT code in $H(m, q)$, with $\delta \geq 5$, containing a block of imprimitivity which is a 1-dimensional vector space. Then $q = 2$ and C is equivalent to the repetition code, the Hadamard code of length 12 or its punctured code.

Alphabet-Affine Codes

This is the final part of my project. What we know:

- The kernel $K_{\mathbf{0}}$ of the action $X_{\mathbf{0}}^{M}$ can be taken to be a straight diagonal subgroup acting semi-regularly on $Q^{\times}$
- An isomorphic copy of $K_{\mathbf{0}}$ must appear as a normal subgroup of the transitive linear group $X_{\mathbf{0},1}^{Q^{\times}}$
- We have used this to show $K_{\mathbf{0}}$ is soluble
- $X_{\mathbf{0},1,2}^{Q^{\times}}$ is also transitive
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- $X_{\mathbf{0},1,2}^{Q^{\times}}$ is also transitive
- C has a block of imrprimitivity which is a vector space

We aim use the interplay between the 2-homogeneous $X_{\mathbf{0}}^{M}$ and the transitive linear groups $X_{\mathbf{0},1}^{Q^{\times}}$ and $X_{\mathbf{0},1,2}^{Q^{\times}}$ to characterise the possible blocks of imprimitivity of C

Elusive codes

For an *s-elusive code*, the group of automorphisms of the s-neighbours is larger than that of the code

Theorem (DH)

Let C be an s-elusive code in $H(m, q)$ with $\delta \geq 2s + 1$. Then there exists a q-ary s-$(m, 2s, 1)$-design.

- certain permutation codes are 1-elusive
- Reed-Muller codes are 1-elusive
- Preparata codes are 2-elusive
- the even weight subcode of the perfect Golay code is 3-elusive

Future directions

Possible future work includes:

- Classify all 2-neighbour transitive codes from blocks
- Explicitly classify all completely transitive codes
- Explore designs from any new codes
- Extend techniques to look ak imprimitive rank 3 groups
- Classify those 2-neighbour transitive codes which are also elusive

Cheers!

Thanks!