Parallel classes in Steiner triple systems

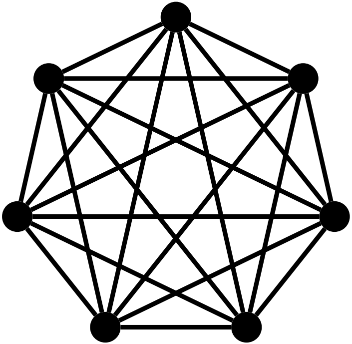
Steiner triple systems

Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.

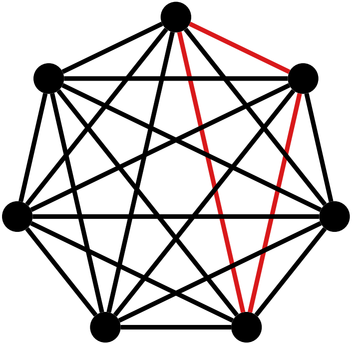
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



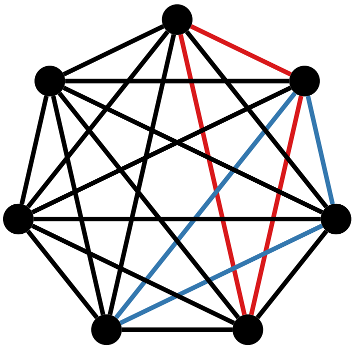
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



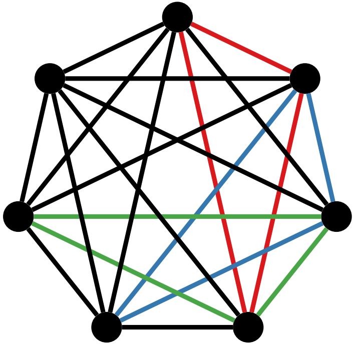
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



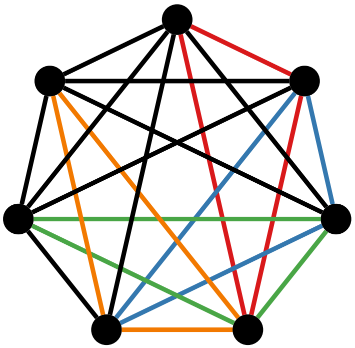
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



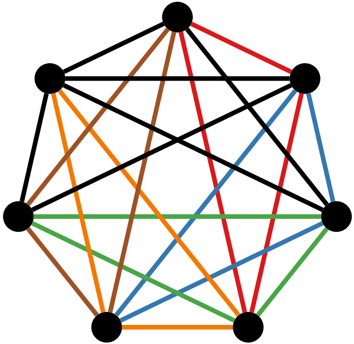
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



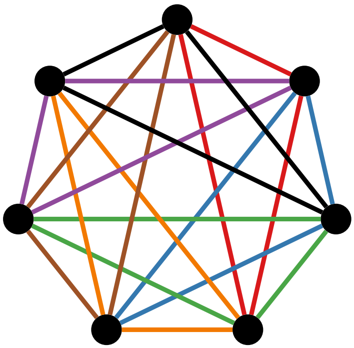
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



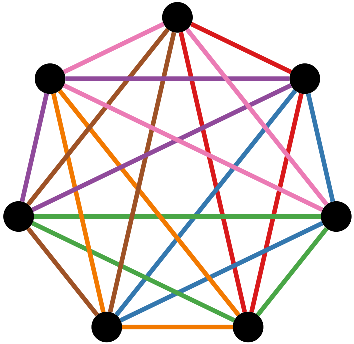
Steiner triple systems

A collection of 3-subsets (*triples*) of a *v*-set (of *points*) such that every pair of points appears together in exactly one triple.



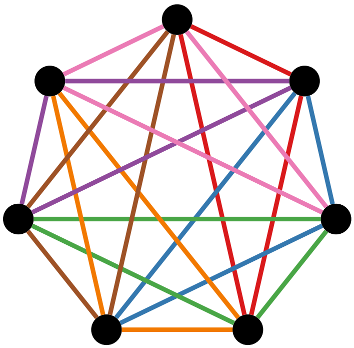
Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.



Steiner triple systems

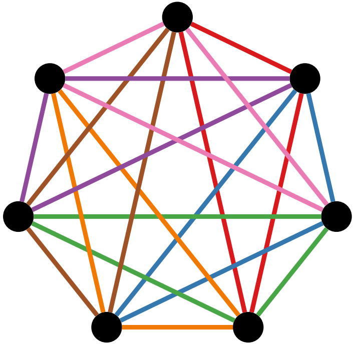
A collection of 3-subsets (*triples*) of a *v*-set (of *points*) such that every pair of points appears together in exactly one triple.



An STS(7)

Steiner triple systems

A collection of 3-subsets (*triples*) of a v-set (of *points*) such that every pair of points appears together in exactly one triple.

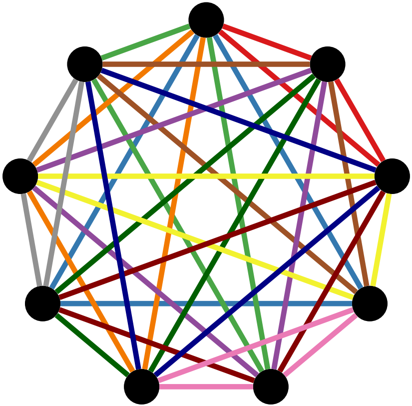


An STS(7)

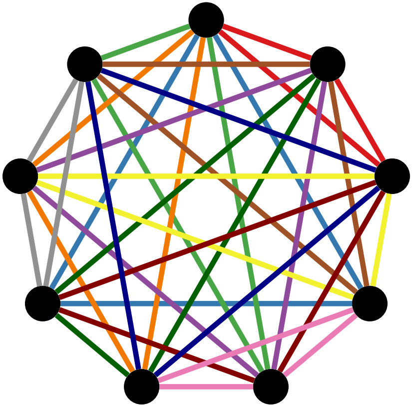
Theorem (Kirkman 1847)
An STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Partial parallel classes

Partial parallel classes

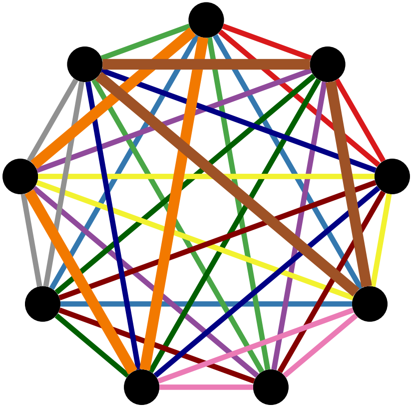


Partial parallel classes



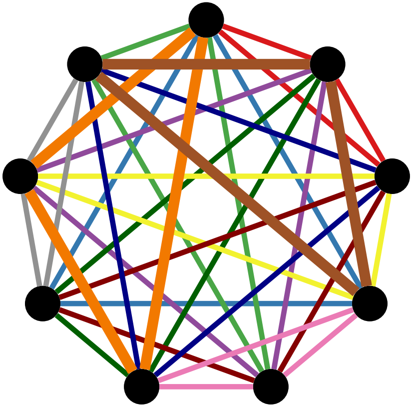
An STS(9)

Partial parallel classes



An STS(9)

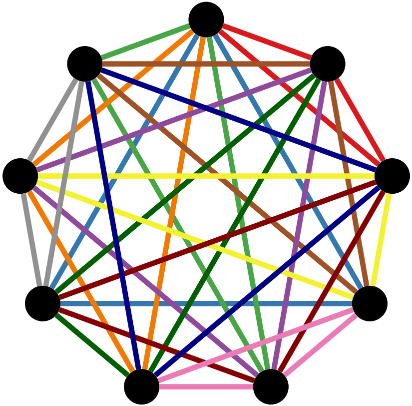
Partial parallel classes



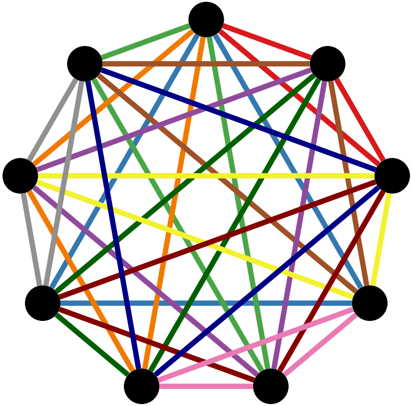
A PPC in the STS(9)

Parallel classes

Parallel classes

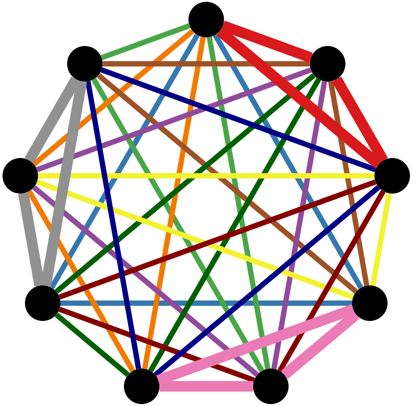


Parallel classes



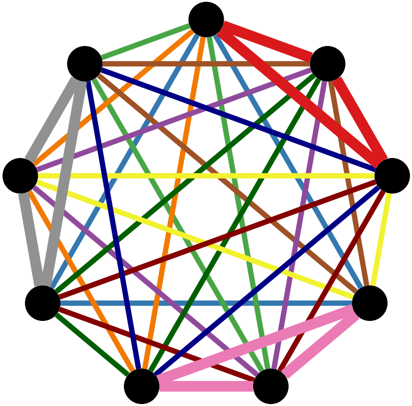
An STS(9)

Parallel classes



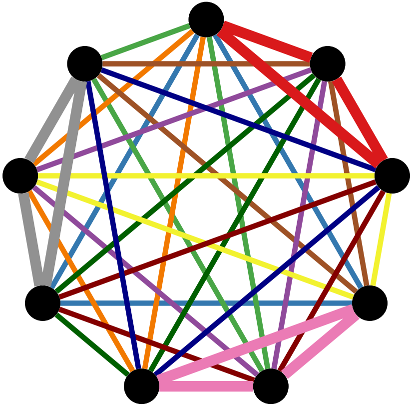
An STS(9)

Parallel classes



A PC in the STS(9)

Parallel classes

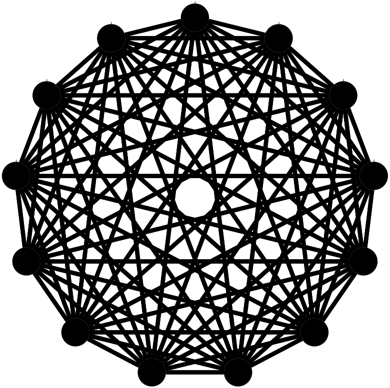


A PC in the STS(9)

STSs with orders 3 (mod 6) are candidates to have PCs.

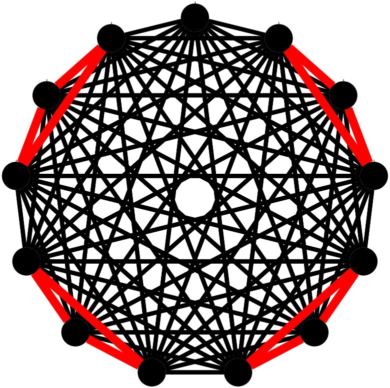
Almost parallel classes

Almost parallel classes



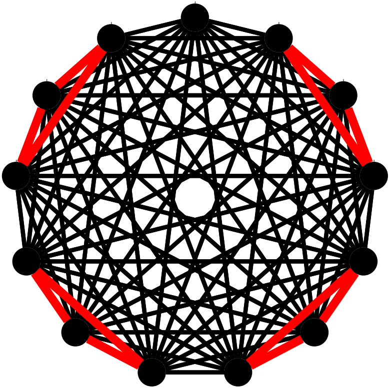
An STS(13)

Almost parallel classes



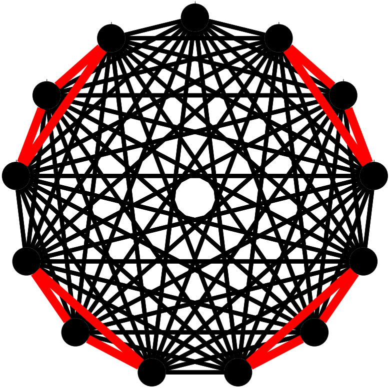
An STS(13)

Almost parallel classes



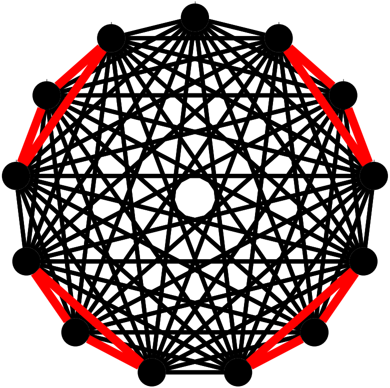
An APC in the STS(13)

Almost parallel classes



An APC in the STS(13)

Almost parallel classes



An APC in the STS(13)

STSs with orders 1 (mod 6) are candidates to have APCs.

Three questions

What can we say about:

1. when an STS has a PC/APC?
2. the size of the largest PPC in an STS?
3. the minimum number of PPCs into which an STS can be partitioned?

Question 1

Question 1

What can we say about when an STS has a PC/APC?

Small orders

Small orders

- The unique STS(7) has no APC.

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.
- All but 10 of the 80 STS(15)s have a PC.

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.
- All but 10 of the 80 STS(15)s have a PC.
- All but 2 of the $11,084,874,829$ STS(19)s have an APC.
  – Colbourn et al.

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.
- All but 10 of the 80 STS(15)s have a PC.
- All but 2 of the $11,084,874,829$ STS(19)s have an APC.
  – Colbourn et al.
- All but 12 of the 1772 *4-rotational* STS(21)s have a PC.
  – Mathon, Rosa

Small orders

- The unique STS(7) has no APC.
- The unique STS(9) has a PC.
- Both STS(13)s have an APC.
- All but 10 of the 80 STS(15)s have a PC.
- All but 2 of the $11,084,874,829$ STS(19)s have an APC.
  – Colbourn et al.
- All but 12 of the 1772 *4-rotational* STS(21)s have a PC.
  – Mathon, Rosa

STSs without PCs/APCs seem rare.

STS with PC/APCs

STS with PC/APCs

Theorem (Lu, Ray-Chaudhuri, Wilson)
For all $v \equiv 3 \pmod 6$, there is an STS(v) whose triples can be partitioned into $\frac{v-1}{2}$ PCs. (*Kirman triple systems*)

STS with PC/APCs

Theorem (Lu, Ray-Chaudhuri, Wilson)
For all $v \equiv 3 \pmod 6$, there is an STS(v) whose triples can be partitioned into $\frac{v-1}{2}$ PCs. (*Kirman triple systems*)

Theorem (Vanstone, Stinson, Schellenberg)
For all $v \equiv 1 \pmod 6$, $v \neq 7$, there an STS(v) whose triples can be partitioned into $\frac{v-1}{2}$ APCs and one other PPC. (*Hanani triple systems*)

STS with PC/APCs

Theorem (Lu, Ray-Chaudhuri, Wilson)
For all $v \equiv 3 \pmod 6$, there is an STS(v) whose triples can be partitioned into $\frac{v-1}{2}$ PCs. (*Kirman triple systems*)

Theorem (Vanstone, Stinson, Schellenberg)
For all $v \equiv 1 \pmod 6$, $v \neq 7$, there an STS(v) whose triples can be partitioned into $\frac{v-1}{2}$ APCs and one other PPC. (*Hanani triple systems*)

It seems the vast majority of STS contain a PC/APC.

STS without PC/APCs

STS without PC/APCs

$v \equiv 1 \pmod{6}$

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS($2(3^n) + 1$) for each $n \geq 1$.

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS($2(3^n) + 1$) for each $n \geq 1$.

$v \equiv 3 \pmod 6$

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS($2(3^n) + 1$) for each $n \geq 1$.

$v \equiv 3 \pmod 6$

Conjecture (Mathon, Rosa)
There is a PC free STS(v) for all $v \equiv 3 \pmod 6$ except $v = 3, 9$.

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS($2(3^n) + 1$) for each $n \geq 1$.

$v \equiv 3 \pmod 6$

Conjecture (Mathon, Rosa)
There is a PC free STS(v) for all $v \equiv 3 \pmod 6$ except $v = 3, 9$.

Theorem (Bryant, Horsley, 2015)
There is a PC free STS($5p + 2$) for each prime $p \equiv 5 \pmod{24}$.

STS without PC/APCs

$v \equiv 1 \pmod 6$

Conjecture (Rosa, Colbourn)
There is an APC free STS(v) for all $v \equiv 1 \pmod 6$ except $v = 13$.

Theorem (Wilson, 1992)
For each odd n there is an APC free STS($2^n - 1$).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS($2(3^n) + 1$) for each $n \geq 1$.

$v \equiv 3 \pmod 6$

Conjecture (Mathon, Rosa)
There is a PC free STS(v) for all $v \equiv 3 \pmod 6$ except $v = 3, 9$.

Theorem (Bryant, Horsley, 2015)
There is a PC free STS($5p + 2$) for each prime $p \equiv 5 \pmod{24}$.

Question 2

Question 2

What can we say about the size of the largest PPC in an STS?

Theorem (Alon, Kim, Spencer, 1995)
Any STS(v) contains a PPC covering all points but at most $O(v^{1/2} \ln^{3/2} v)$.

Theorem (Alon, Kim, Spencer, 1995)
Any STS(v) contains a PPC covering all points but at most $O(v^{1/2} \ln^{3/2} v)$.

No example is known of an STS(v) whose largest PPC covers fewer than $v - 4$ points.

Question 3

Question 3

What can we say about the minimum number of PPCs into which an STS can be partitioned?

Question 3

What can we say about the minimum number of PPCs into which an STS can be partitioned?

This number is called the *chromatic index* of the system.

Bounds on chromatic indices

Bounds on chromatic indices

The chromatic index of any STS(v) is at least

$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod 6 \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod 6 \end{cases}$$

Bounds on chromatic indices

The chromatic index of any STS(v) is at least

$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod 6 \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod 6 \end{cases}$$

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Bounds on chromatic indices

The chromatic index of any STS(v) is at least

$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod 6 \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod 6 \end{cases}$$

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Result (Colbourn, Colbourn, 1982)
Any STS(v) has chromatic index at most $\frac{3v-3}{2}$.

Bounds on chromatic indices

The chromatic index of any STS(v) is at least

$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod 6 \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod 6 \end{cases}$$

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Result (Colbourn, Colbourn, 1982)
Any STS(v) has chromatic index at most $\frac{3v-3}{2}$.

Theorem (Pippenger, Spencer, 1989)
The maximum chromatic index over the STS(v)s approaches $\frac{v}{2}$ as $v \to \infty$.

Bounds on chromatic indices

The chromatic index of any STS(v) is at least

$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod 6 \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod 6 \end{cases}$$

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Result (Colbourn, Colbourn, 1982)
Any STS(v) has chromatic index at most $\frac{3v-3}{2}$.

Theorem (Pippenger, Spencer, 1989)
The maximum chromatic index over the STS(v)s approaches $\frac{v}{2}$ as $v \to \infty$.

Systems with low/high chromatic indices

Systems with low/high chromatic indices

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Systems with low/high chromatic indices

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Kirkman and Hanani triple systems supply examples of STS(v)s with chromatic index $m(v)$ for all $v \neq 1, 7$.

Systems with low/high chromatic indices

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Kirkman and Hanani triple systems supply examples of STS(v)s with chromatic index $m(v)$ for all $v \neq 1, 7$.

PC/APC free STS necessarily have chromatic index at least $m(v)+2$.

Systems with low/high chromatic indices

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Kirkman and Hanani triple systems supply examples of STS(v)s with chromatic index $m(v)$ for all $v \neq 1, 7$.

PC/APC free STS necessarily have chromatic index at least $m(v)+2$.

Theorem (Bryant, Colbourn, Horsley, Wanless)
For all $v \equiv 3 \pmod 6$ there is a STS(v) with chromatic index at least $m(v)+2$ except when $v \in \{3, 9\}$ and possibly when $v \in \{45, 75, 129, 513\}$.

Systems with low/high chromatic indices

Conjecture (Rosa)
Every STS(v) has chromatic index in $\{m(v), m(v)+1, m(v)+2\}$.

Kirkman and Hanani triple systems supply examples of STS(v)s with chromatic index $m(v)$ for all $v \neq 1, 7$.

PC/APC free STS necessarily have chromatic index at least $m(v)+2$.

Theorem (Bryant, Colbourn, Horsley, Wanless)
For all $v \equiv 3 \pmod 6$ there is a STS(v) with chromatic index at least $m(v)+2$ except when $v \in \{3, 9\}$ and possibly when $v \in \{45, 75, 129, 513\}$.

Theorem (Baker, Meszka)
For $n \geq 3$, the chromatic index of a projective STS($2^n - 1$) is $m(2^n-1)+2$ if n is odd and is $m(2^n-1)$ if n is even.

Thanks.