Parallel classes in Steiner triple systems
Steiner triple systems

A collection of 3-subsets (triples) of a \(v\)-set (of points) such that every pair of points appears together in exactly one triple.

Theorem (Kirkman 1847)

An STS(\(v\)) exists if and only if \(v \equiv 1 \text{ or } 3 \pmod{6}\).
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An STS(7)
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**Theorem** *(Kirkman 1847)*

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Partial parallel classes
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An STS(9)
Partial parallel classes

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A PPC in the STS(9)
Parallel classes

An STS(9) A PC in the STS(9) STSs with orders 3 (mod 6) are candidates to have PCs.
Parallel classes

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STSs with orders 3 (mod 6) are candidates to have PCs.
Almost parallel classes
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An STS(13)

An APC in the STS(13)

STSs with orders 1 (mod 6) are candidates to have APCs.
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STSs with orders 1 (mod 6) are candidates to have APCs.
Three questions

What can we say about:

1. when an STS has a PC/APC?

2. the size of the largest PPC in an STS?

3. the minimum number of PPCs into which an STS can be partitioned?
Question 1
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What can we say about when an STS has a PC/APC?
Small orders

The unique STS (7) has no APC.

The unique STS (9) has a PC.

Both STS (13)s have an APC.

All but 10 of the 80 STS (15)s have a PC.

All but 2 of the 11,084,874 STS (19)s have an APC.

– Colbourn et al.

All but 12 of the 1772 4-rotational STS (21)s have a PC.

– Mathon, Rosa

STSs without PCs/APCs seem rare.
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STSs without PCs/APCs seem rare.
Theorem (Lu, Ray-Chaudhuri, Wilson)
For all \( v \equiv 3 \pmod{6} \), there is an STS \((v)\) whose triples can be partitioned into \( v - 1 \) 2 PC\( s \). (Kirman triple systems)

Theorem (Vanstone, Stinson, Schellenberg)
For all \( v \equiv 1 \pmod{6} \), \( v \neq 7 \), there is an STS \((v)\) whose triples can be partitioned into \( v - 1 \) 2 APCs and one other PPC. (Hanani triple systems)

It seems the vast majority of STS contain a PC/APC.
Theorem (Lu, Ray-Chaudhuri, Wilson)
For all \( v \equiv 3 \pmod{6} \), there is an STS\((v)\) whose triples can be partitioned into \( \frac{v-1}{2} \) PCs. (*Kirman triple systems*)

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STS with PC/APCs

**Theorem** (Lu, Ray-Chaudhuri, Wilson)  
For all $v \equiv 3 \pmod{6}$, there is an STS($v$) whose triples can be partitioned into $\frac{v-1}{2}$ PCs. (*Kirman triple systems*)

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STS without PC/APCs

Conjecture (Rosa, Colbourn)
There is an APC free STS \((v)\) for all \(v \equiv 1 \pmod{6}\) except \(v = 13\).

Theorem (Wilson, 1992)
For each odd \(n\) there is an APC free STS \((2^{n} - 1)\).

Theorem (Bryant, Horsley, 2013)
There is an APC free STS \(\left(2 \left(3^n\right) + 1\right)\) for each \(n \geq 1\).

Conjecture (Mathon, Rosa)
There is a PC free STS \((v)\) for all \(v \equiv 3 \pmod{6}\) except \(v = 3, 9\).

Theorem (Bryant, Horsley, 2015)
There is a PC free STS \(\left(5p + 2\right)\) for each prime \(p \equiv 5 \pmod{24}\).
STS without PC/APCs

\[ v \equiv 1 \pmod{6} \]
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Question 2
What can we say about the size of the largest PPC in an STS?
Theorem (Alon, Kim, Spencer, 1995)

Any STS \((v)\) contains a PPC covering all points but at most \(O\left(\frac{v}{2} \ln \frac{3}{2} v\right)\).

No example is known of an STS \((v)\) whose largest PPC covers fewer than \(v - 4\) points.
**Theorem** (Alon, Kim, Spencer, 1995)

Any STS\( (v) \) contains a PPC covering all points but at most \( O(v^{1/2} \ln^{3/2} v) \).
Theorem (Alon, Kim, Spencer, 1995)
Any STS($v$) contains a PPC covering all points but at most $O(v^{1/2} \ln^{3/2} v)$.

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Question 3

What can we say about the minimum number of PPCs into which an STS can be partitioned? This number is called the chromatic index of the system.
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This number is called the *chromatic index* of the system.
Bounds on chromatic indices

The chromatic index of any STS \((v)\) is at least

\[
m(v) = \begin{cases} 
  v - 1, & \text{if } v \equiv 3 \pmod{6} \\
  v + 1, & \text{if } v \equiv 1 \pmod{6}
\end{cases}
\]

Conjecture (Rosa) Every STS \((v)\) has chromatic index in \([m(v), m(v) + 1, m(v) + 2]\).

Result (Colbourn, Colbourn, 1982) Any STS \((v)\) has chromatic index at most \(3v - 3\).

Theorem (Pippenger, Spencer, 1989) The maximum chromatic index over the STS \((v)\) approaches \(v^2\) as \(v \to \infty\).
Bounds on chromatic indices

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$$m(v) = \begin{cases} \frac{v-1}{2}, & \text{if } v \equiv 3 \pmod{6} \\ \frac{v+1}{2}, & \text{if } v \equiv 1 \pmod{6} \end{cases}$$
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Systems with low/high chromatic indices

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Kirkman and Hanani triple systems supply examples of STS \((v)\)s with chromatic index \(m(v)\) for all \(v \neq 1, 7\).

PC/APC free STS necessarily have chromatic index at least \(m(v) + 2\).

Theorem (Bryant, Colbourn, Horsley, Wanless)
For all \(v \equiv 3\) (mod 6) there is a STS \((v)\) with chromatic index at least \(m(v) + 2\) except when \(v \in \{3, 9\}\) and possibly when \(v \in \{45, 75, 129, 513\}\).

Theorem (Baker, Meszka)
For \(n \geq 3\), the chromatic index of a projective STS \((2^n - 1)\) is \(m(2^n - 1) + 2\) if \(n\) is odd and is \(m(2^n - 1)\) if \(n\) is even.
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