Tutte Polynomials, Flow Polynomials and Chromatic Polynomials

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September 2016

COLLABORATORS

Much of this work was stimulated by the paper[†]

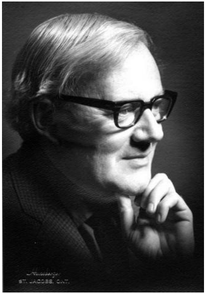
TUTTE RELATIONS, TQFT AND PLANARITY OF CUBIC GRAPHS

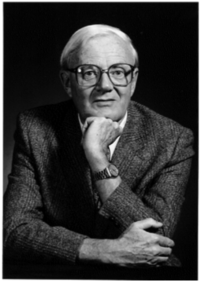
`arxiv:1512.07339`

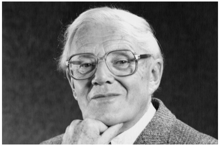
and subsequent (email) conversations with its authors Ian Agol (Berkeley/Princeton) and Slava Krushkal (Virginia).

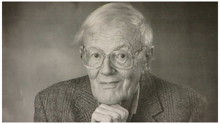
[†] TQFT = Topological Quantum Field Theory

Tutte









Tutte polynomial

The Tutte polynomial can be defined *recursively* such that $T_G(x,y) = 1$ when G has no edges, and

$$T_G(x,y) = \begin{cases} x\,T_{G/e}(x,y), & \text{if } e \text{ is a bridge;} \\ y\,T_{G\backslash e}(x,y), & \text{if } e \text{ is a loop;} \\ T_{G\backslash e}(x,y) + T_{G/e}(x,y), & \text{otherwise.} \end{cases}$$

Here $G\backslash e$ is the graph obtained by *deleting* e and G/e the graph obtained by *contracting* e.

It is a surprising result, due to Tutte, that the result is independent of the order in which edges are chosen.

TUTTE POLYNOMIAL FOR PETERSEN

	y^0	y^1	y^2	y^3	y^4	y^5	y^6
x^0	0	36	84	75	35	9	1
x^1	36	168	171	65	10		
x^2	120	240	105	15			
x^3	180	170	30				
x^4	170	70					
x^5	114	12					
x^6	56						
x^7	21						
x^8	6						
x^9	1						

Two relevant polynomials

For a connected graph G, the *chromatic polynomial* is

$$P_G(q) = (-1)^{|V(G)|-1} q T_G(1-q, 0)$$

and the *flow polynomial* is

$$F_G(q) = (-1)^{|E(G)|-|V(G)|+1} T_G(0, 1-q).$$

For a positive integer q,

- $P_G(q)$ is the number of proper q-colourings of G, and
- $F_G(q)$ is the number of nowhere-zero $\mathbb{Z}_q$-flows of G

Duality

If G is planar, then it has a *planar dual* G^* and

$$T_{G^*}(x, y) = T_G(y, x)$$

As a consequence, if G is connected, then

$$F_{G^*}(q) = q^{-1} P_G(q).$$

THE BIG PICTURE

Two classes of graphs play a major role in the study of graph polynomials and in graph theory overall.

- Let $\mathcal{T}$ denote the set of *planar triangulations*
  Graphs in $\mathcal{T}$ are maximal planar, have only triangular faces, and have $3V - 6$ edges
- Let $\mathcal{C}$ denote the set of *planar cubic graphs*
  Graphs in $\mathcal{C}$ are regular of degree 3, have $3V/2$ edges and $(V + 4)/2$ faces

The classes $\mathcal{T}$ and $\mathcal{C}$ are dual to each other.

What are we studying?

Generic question *What properties of graphs are reflected in their Tutte, flow and/or chromatic polynomials?*

Today we'll consider a few specialisations of this generic question:

Today's question(s)

What — if any — properties of the Tutte, chromatic and/or flow polynomials certify membership in $\mathcal{T}$ (among all graphs) or membership in $\mathcal{C}$ (among all cubic graphs)?

Outline

- Golden Identity
- Dual Golden Identity
- Parity

TUTTE'S GOLDEN IDENTITY

If T is a *planar triangulation* on n vertices, then

$$P_T(\varphi + 2) = (\varphi + 2)\varphi^{3n-10} P_T(\varphi + 1)^2,$$

where $\varphi = (1 + \sqrt{5})/2$ is the Golden Ratio.

$$P(T, \tau\sqrt{5}) = \sqrt{5} \cdot \tau^{3(k-3)} \cdot P^2(T, \tau^2). \qquad (11.12)$$

This result was published in [105]. The proof given there is marred by a fallacious extension beyond the realm of triangulations. But for triangulations it is valid.

Formula (11.12) is called the Golden Identity. Sometimes people find it difficult to believe at first. Dick Wick Hall told me he had not done so

Golden Graphs

Are there any *golden graphs*† other than planar triangulations?

† temporary notation

GOLDEN GRAPHS

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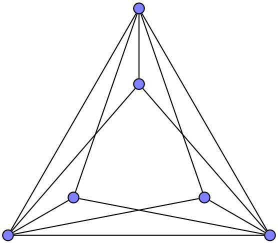


It is easy to see that

$$P(G,q) = q(q-1)(q-2)(q-3)^3$$

just by counting q-colourings.

† temporary notation

Golden Polynomials

Proposition

If $P(q)$ satisfies the Golden Identity, then so does

$$(q-2)P(q) \text{ and also } (q-3)P(q).$$

If $Q(q) = (q-2)P(q)$ then

$$Q(\varphi+1) = (\varphi-1)P(\varphi+1)$$

and so

$$\varphi^2 Q(\varphi+1)^2 = P(\varphi+1)^2$$

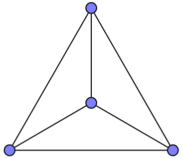
and then simplify in the obvious fashion.

All polynomials with shape $q(q-1)(q-2)^a(q-3)^b$ are golden.

Simplicial vertices

A vertex is *simplicial* if its neighbourhood is a clique:

- Start with any planar triangulation T
- Repeatedly add simplicial vertices of degrees 2 and 3



These graphs are *chordal* if T is a clique.

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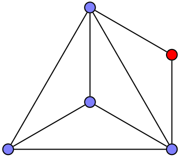

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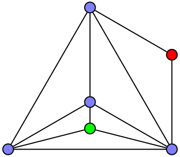

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SMALL CONNECTED GOLDEN GRAPHS

For 4–11 vertices, there are modest — but growing — numbers.[†]

n	$g(n)$
4	2
5	4
6	15
7	61
8	357
9	2755
10	27519
11	328847

[†] Nope, no matching numbers in the OEIS.

ANALYSIS

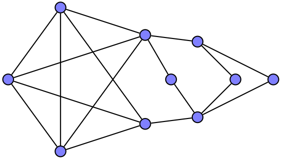
The first *non-chordal* graph arises at 6 vertices — it is a planar graph, namely the *octahedron*, with chromatic polynomial

$$(q-2)(q-1)q(q^3-9q^2+29q-32).$$

The first *non-planar* graph also arises at 6 vertices, and is the chordal graph K_3+3K_1 shown above, where $+$ denotes the *complete join* of its arguments.

LIKELY TO BE HARD

Small (2- or even 1-vertex) cutsets abound, and if G is



$$P_G(q) = (q-3)(q-1)q(q-2)^2(q^5 - 9q^4 + 34q^3 - 71q^2 + 83q - 41)$$

which satisfies

$$\frac{P_G(\varphi+2)}{(\varphi+2)\varphi^{3n-10}P_G(q+1)^2} = 2+\sqrt{5}$$

implying that $(q-1)P_G(q)$ is golden.

Minor aside

Chordal graphs have *integral* chromatic roots – but not all graphs with integral chromatic roots are chordal.

Conjecture (Dong)

A graph with all chromatic roots in the set $\{0, 1, 2, 3\}$ *is chordal.*

Dong has proved this is true for *planar* graphs.

Joe Kung & I proved that graphs with *integral flow roots* are dual planar chordal and Dong conjectures that the hypothesis can be weakened to *real flow roots*.

Outline

Golden Identity

Dual Golden Identity

Parity

Tutte's Dual Golden Identity

If G is a *planar cubic graph* then its flow polynomial $F_G(x)$ satisfies

$$F_G(\varphi+2) = \varphi^{|E(G)|} F_G(\varphi+1)^2.$$

$$\begin{aligned}
F_G(\varphi+2) &= (\varphi+2)^{-1} P_{G^*}(\varphi+2) \\
&= (\varphi+2)^{-1} \underbrace{(\varphi+2)\varphi^{3(n+4)/2-10} P_{G^*}(\varphi+1)^2}_{\text{golden identity for } G^*} \\
&= \varphi^{3(n+4)/2-10} \underbrace{(\varphi+1)^2 F_G(\varphi+1)^2}_{\text{dual chromatic}} \\
&= \varphi^{3(n+4)/2-10} \varphi^4 F_G(\varphi+1)^2 \\
&= \varphi^{3n/2} F_G(\varphi+1)^2 \\
&= \varphi^{|E(G)|} F_G(\varphi+1)^2
\end{aligned}$$

Non-planar?

Conjecture (Agol & Krushkal)

If G is a ~~planar~~ *cubic graph* then its flow polynomial $F_G(x)$ satisfies

$$|F_G(\varphi + 2)| \leq \varphi^{|E(G)|} F_G(\varphi + 1)^2.$$

Moreover, equality holds in this inequation if and only if G is planar.

Is this plausible?

On 14 vertices, we have $\phi^{21} \approx 24476$ and so the ratio

$$\frac{|F_G(\varphi + 2)|}{F_G(\varphi + 1)^2} \leq 24476$$

with equality for the planar graphs.

For the remaining cubic graphs, this ratio ranges from

$$0.00166937 \text{ to } 1453.5556.$$

The conjecture holds for cubic graphs on up to 24 vertices — by 24 vertices there are 117940535 cubic graphs having been tested.

Outline

Golden Identity

Dual Golden Identity

Parity

Let's get weird

What is the distribution of the *constant term* of the flow polynomial of a cubic graph taken mod 5?

n	0	1	2	3	4
10	8	-	7	4	-
12	29	27	-	-	29
14	199	-	145	165	-
16	1489	1307	-	-	1264
18	14723	-	13557	13021	-
20	172341	167428	-	-	170720

TRUE FOR PLANAR

This can be proved for *planar* cubic graphs.

Working modulo $\sqrt{5}$ in $\mathbb{Q}[\sqrt{5}]$ † we have

$$\varphi + 2 = (5 + \sqrt{5})/2 = 0$$

and so $\varphi + 1 = 4$ and $\varphi = 3$.

Therefore Tutte's Golden Identity (dual version) gives

$$F_G(0) = 3^{|E(G)|} F_G(4)^2$$

and as these are all integers, this mod $\sqrt{5}$ congruence must actually hold mod 5.

† In this field the "integers" are the natural integer linear combinations of 1 and φ

What about the rest?

Does "*this pattern*" hold for all cubic graphs, and if so, why?

Narrowing down to a precise question, suppose that G is a bridgeless cubic graph with *chromatic index* † equal to 4.

Then $F_G(4) = 0$ — but is it true that $5 \mid F_G(0)$?

(As F_G has a factor of $(q-1)(q-2)(q-3)(q-4)$, it is immediate that $24 \mid F_G(0)$)

† i.e. colouring edges rather than the vertices

Conjecture

Wild Conjecture

If G is a cubic graph that is not 3-edge colourable, then $240 \mid F_G(0)$.

Of course $F_G(0) = T_G(1,0)$, and $T_G(1,0)$ has a combinatorial interpretation:

Proposition (Chen, Stanley, Gioan, ?)

$T_G(1,0)$ is the number of distinct *indegree sequences* of the *totally cyclic orientations* of G.

PICTURES

G_1, G_2, G_3, G_4 are locally related as follows (and are identical otherwise)

$G_1 =$  $G_2 =$  $G_3 =$  $G_4 =$ 

Then

$$F_{G_1}(0) + F_{G_2}(0) + 2F_{G_3}(0) + F_{G_4}(0) = 0$$


תודה
Dankie
Gracias
Спасибо
شكراً
Merci
Takk
Köszönjük
Terima kasih
Grazie
Dziękujemy
Dėkojame
Ďakujeme
Vielen Dank
Paldies
Kiitos
Täname teid
谢谢
Thank You
Tak
感謝您
Obrigado
Teşekkür Ederiz
감사합니다
Σας ευχαριστούμε
ขอบคุณ
Bedankt
Děkujeme vám
ありがとうございます
Tack