Planes and MOLS

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A few of our favourite things

- A projective plane of order n.
- An orthogonal array $\text{OA}(n+1, n)$ (strength 2)
- A (complete) set of $n-1$ $\text{MOLS}(n)$.

A classical result says these objects are "equivalent".

But how well do we understand the relationship?

An analogous situation existed for symmetric idempotent Latin squares and 1-factorisations of complete graphs.

Plane to OA

Given a PP π or order n, choose a line L and use it to construct $A(\pi, L)$, an OA$(n+1, n)$.

- Label the points on L as $P_1, \ldots, P_{n+1}$.
- For $1 \leqslant i \leqslant n+1$, number the lines (other than L) through P_i as $M_{i1}, \ldots, M_{in}$;
- Label the points not on L as $Q_1, \ldots, Q_{n^2}$;
- now $A(\pi, L)$ is the $n^2 \times (n+1)$ array with (i, j) entry k if the line through Q_i and P_j is M_{jk}.

The choice of L matters! (Other choices don't).

The number of inequivalent OAs obtainable from π is equal to the number of orbits of the collineation group of π on lines.

Symmetries

Each object has a notion of symmetry but how do symmetries of one object manifest as symmetries of another?

For planes we have the collineation group (permutations of points that preserve lines).

For OAs we have an automorphism group which is the stabiliser in $(\mathcal{S}_n \wr \mathcal{S}_{n+1}) \times \mathcal{S}_{n^2}$

Is the setwise stabiliser of L (the line at infinity) isomorphic to the automorphism group of $A(\pi, L)$?

Parity of Latin squares

The parity of a permutation is either even or odd, depending on whether it can be written as a product of an even or odd number of transpositions.

A *latin square* of order n is an $n \times n$ matrix in which each of n symbols occurs exactly once in each row and once in each column.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ is a latin square of order 4.

Each row is a permutation, so it has a parity.
Each column is a permutation, so it has a parity.
Each symbol also has a parity.

Parity of Latin squares

Let $\sigma : \mathcal{S}_n \mapsto \mathbb{Z}_2$ be the sign homomorphism.

Define $\pi_r = \sum_i \sigma(\text{row } i)$ to be the *row parity*.

Define $\pi_c = \sum_j \sigma(\text{column } j)$ to be the *column parity*.

Define $\pi_s = \sum_k \sigma(\text{symbol } k)$ to be the *symbol parity*.

$$\pi_r \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{aligned} &= \sigma(1234) + \sigma(2413) + \sigma(3142) + \sigma(4321) \\ &= 0 + 1 + 1 + 0 = 0 \mod 2. \end{aligned}$$

$$\pi_s \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \sigma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sigma \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$+\sigma \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \sigma \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 1 + 0 + 0 + 1 = 0 \mod 2$$

The fundamental theorem of Latin square parities

Theorem: $\pi_r + \pi_c + \pi_s \equiv \begin{cases} 0 & \text{if } n = 0, 1 \mod 4, \\ 1 & \text{if } n = 2, 3 \mod 4. \end{cases}$

This theorem has been proved by Janssen ['95], Wanless ['04], Derksen [unpub.], Donovan et al. ['10], and by us again!

So for Latin squares of a given order there are 4 possible parities $\pi_r\pi_c\pi_s$. i.e. 2 parity 'bits'

For $n \equiv 0, 1 \mod 4$ we have $\pi_r\pi_c\pi_s \in \{000, 011, 101, 110\}$.
For $n \equiv 2, 3 \mod 4$ we have $\pi_r\pi_c\pi_s \in \{001, 010, 100, 111\}$.

Parity of MOLS

To develop a concept of parity for MOLS it makes sense to consider the corresponding orthogonal array.

1	1	1	1
1	2	2	2
1	3	3	3
2	1	2	3
2	2	3	1
2	3	1	2
3	1	3	2
3	2	1	3
3	3	2	1

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

A set of k-MOLS(n) is equivalent to a $n^2 \times (k+2)$ orthogonal array, where each pair of columns contains each ordered pair in $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

Our Parity of MOLS

Any 3 columns of the orthogonal array defines a Latin square, which therefore has 3 parities (which are related by the fundamental theorem).

So for every ordered triple of columns we have a parity (either 0 or 1).

e.g.

1	1	1	1
1	2	2	2
1	3	3	3
2	1	2	3
2	2	3	1
2	3	1	2
3	1	3	2
3	2	1	3
3	3	2	1

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma(\varepsilon) + \sigma((231)) + \sigma((321)) = 0 + 0 + 0 = 0$$

The *parity* of an OA, is the function from triples of columns to $\mathbb{Z}_2$.

Basic properties

Let τ_{ij}^c denote the parity obtained by using column c to partition the rows of the OA into n sets of size n, then considering the permutations mapping from column i to column j.

I'll call column c the reference column.

Our parity is invariant under permuting the rows.

Permuting symbols in the reference column has no effect.

If n is odd, we invert the parity τ_{ij}^c by applying a transposition to the symbols in either of the columns i, j.

$\tau_{ij}^c = \tau_{ji}^c$, so our parity is undirected.

How many parity bits?

By composition of permutations, $\tau_{ij}^c + \tau_{jk}^c = \tau_{ik}^c$.

Also, by the fundamental theorem we know that

$$\tau_{bc}^a + \tau_{ca}^b + \tau_{ab}^c \equiv \begin{cases} 0 & \text{if } n = 0,1 \mod 4, \\ 1 & \text{if } n = 2,3 \mod 4. \end{cases}$$

If we know $\{\tau_{2j}^1 : 3 \leqslant j \leqslant k\}$ and $\{\tau_{1j}^c : 3 \leqslant c \leqslant k \text{ and } 2 \leqslant j \leqslant c-1\}$ then all other τ's can be determined.

It follows that an OA(k, n) can have no more than $\binom{k}{2} - 1$ parity bits.

This bound is achieved in many cases including when $k \leqslant 5$ and $n \geqslant 11$ satisfies $n \equiv 3 \mod 4$.

Projective planes

For projective planes there is a further restriction.

A sharply 2-transitive set of permutations is a set of permutations such that for any $(w, x), (y, z)$ ordered pairs of distinct elements there is a *unique* permutation mapping $w \mapsto y$ and $x \mapsto z$.

If $k = n + 1$ then $\{\tau_{12}^{c} : 3 \leqslant c \leqslant k\}$ are the parities of a sharply 2-transitive set of permutations. This is necessarily

$$\binom{n}{2}^2 \equiv \binom{n}{2} \equiv \begin{cases} 0 & \text{if } n = 0, 1 \mod 4, \\ 1 & \text{if } n = 2, 3 \mod 4. \end{cases}$$

We thus reduce the upper bound on the number of parity bits that a projective plane can have. Here we don't have any real idea how many bits there are in practice.

Case study: Order 16

Glynn/Byatt'12 studied the projective planes of order 16.

There are 22 known projective planes of order 16. Each one has potentially 273 different OA(17,16) that it corresponds to.

For 18 of the planes, every parity of every OA is even.

The slightly interesting ones are the Mathon Plane, Johnson Plane, BBS4 and its dual. Even they produce many "all-even" OA's.

Graphical representation

For each $OA(k,n)$ there is a sequence of k graphs, where the i-th graph has vertices $\{1,2,\ldots,k\} \setminus \{i\}$ and has an edge from u to v iff τ^i_{uv} is odd.

Theorem: Each graph is $K_{a,k-1-a}$ for some $0 \leqslant a \leqslant k-1$.

If we take the mod 2 merger of all these parity graphs we get

- $K_{a,k-a}$ if $n \equiv 0,1 \mod 4$.
- $K_a \cup K_{k-a}$ if $n \equiv 2,3 \mod 4$.

(For complete MOLS $a = 0$, so we get the empty graph and complete graph respectively).

Why is 2 mod 4 harder for MOLS?

For prime power $n = 0 \mod 4$ there is an $OA(n+1, n)$ in which the LS that you get from choosing 3 columns are isotopic to each other (and to EA, the elementary abelian group table).

However, if $n \equiv 2 \mod 4$, then any $OA(4, n)$ produces LS from at least 3 different isotopism classes.

We can also show for $n \equiv 2 \mod 8$ that in an $OA(n+1, n)$ all 4 possible parities are achieved by suitable choice of 3 columns.

A botany expedition

OA(17,16) such that any 3 columns give a group table: This only happens for the translation planes; every LS is elementary abelian (EA).

Weaker property: Complete sets of MOLS(16) in which every LS is isotopic to a group:

There are 79 isotopy classes of such MOLS (24 species). They come from 15 different planes. Every translation plane has one (which we expected). Every dual translation plane has one (is there a theorem to prove there?)

The only non-translation planes with one are the Mathon plane and its dual. (To be explained).

For sets in which not every LS is a group table the record is 7 group tables (EA).

There is a set of MOLS having 1 EA plus 14 copies of $Q_8 \times C_2$. It comes from the dual of the Lorimer-Rahilly plane. It is in the same species as two (isotopy classes of) complete sets of MOLS in which every LS is EA.

Open Questions?

- Has anyone written down the connection between orbits on lines and number of inequivalent OAs?
- Has anyone written down the connection between collineation group of π and the automorphism group of $OA(\pi, L)$?
- How many parity bits can a PP have?
- Why are so many complete sets of MOLS all even?
- If every 3 columns in an OA$(n+1, n)$ give EA, is the plane necessarily a translation plane?
- For every dual translation plane is there a corresponding complete set of MOLS that contain only EA?
- Which non-TP and non-dual-TP produce complete sets of MOLS that contain only EA?
- Why does the dual Lorimer-Rahilly plane produce a complete set of MOLS which are all groups, but only one is EA?
- Which groups can be in complete sets of MOLS?