Finite geometry

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Geometry

Analytic Geometry

- Descartes (1596-1650), Fermat (1601-1665)
- Equations to describe curves

Transformation Geometry / Symmetry

- Klein (1849-1925), *Erlanger Programm*, Cayley (1825-1895)
- Study of geometries via their symmetries

Synthetic Geometry

- Since antiquity
- Spatial intuition is used to "construct" geometrical objects

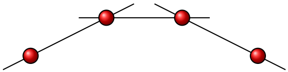
> *Synthetic geometry is that which studies figures as such, without recourse to formulas, whereas analytic geometry consistently makes use of such formulas as can be written down after the adoption of an appropriate system of coordinates.*
>
> *– Felix Klein*

Example: Elliptic quadrics of 3-dim. projective spaces

Analytic
Zeros of a quadratic in four variables: $x_0x_1 + x_2^2 + x_2x_3 + \eta x_3^2 = 0$

Symmetric
Orbit of $\mathrm{P}\Omega^-(4, q)$ on totally singular 1-spaces.

Synthetic
$q^2 + 1$ points of projective space, no three collinear.



Parameters of finite projective planes

Definition (Order)

Let $\mathbb{P}$ be a finite projective plane. Then $\exists n \in \mathbb{N}$ such that there are exactly $n+1$ points on any line and $n+1$ lines through any point.

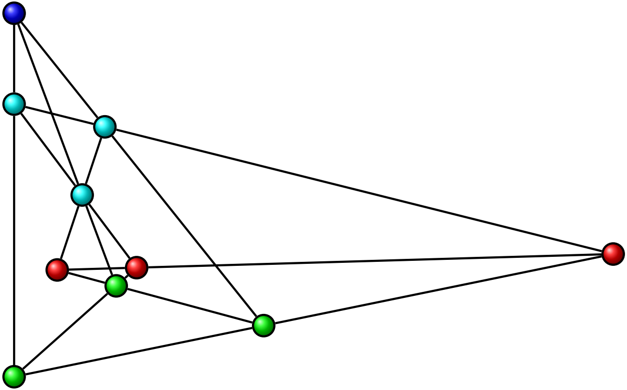
The Prime Power Conjecture

n is a prime power.

$$\begin{aligned} \text{Bruck-Ryser (1949):} \quad & n \equiv 1, 2 \pmod 4 \Rightarrow n = x^2 + y^2 \\ \Rightarrow \quad & n \not\equiv 6 \pmod 8 \\ \text{Bose (1938):} \quad & n \neq 6. \\ \text{Lam, Thiel, Swiercz (1989):} \quad & n \neq 10. \end{aligned}$$

Prime order projective planes

n prime $\Rightarrow$ Desarguesian;
i.e., arises from the lattice of subspaces of $\mathbb{F}_p^3$.



Symmetries

Let Π be a finite projective plane.

Theorem (Ostrom & Wagner 1959)

$\mathrm{Aut}(\Pi)$ *doubly-transitive on points* $\Rightarrow \Pi$ *is Desarguesian.*

Conjecture (D. Hughes 1959)

$\mathrm{Aut}(\Pi)$ transitive on points $\Rightarrow \Pi$ is Desarguesian.

Theorem (Gill 2016)

$G \leqslant \mathrm{Aut}(\Pi)$, *$G$ transitive on points,* Π *non-Desarguesian.*
G insoluble $\Rightarrow G/O(G)$ *is isomorphic to* $\mathrm{SL}(2,5)$ *or* $\mathrm{SL}(2,5).2$.

Ovoids of 3-dimensional projective space

Two known families of examples

	Coordinates	Group
Elliptic quadrics	$(s, t, s^2 + st + \eta t^2, 1)$, $s, t \in \mathbb{F}_q$, $(0, 0, 1, 0)$	${}^2D_2(q)$
Suzuki-Tits ovoids	$(s, t, s^{\sigma+2} + st + t^{\sigma}, 1)$, $s, t \in \mathbb{F}_{2^{2e+1}}$, $(0, 0, 1, 0)$	${}^2B_2(q)$

- q odd $\Rightarrow$ elliptic quadric (Barlotti, Panella 1955),
- Classified for $q \leqslant 32$ (O'Keefe, Penttila & Royle 1994).

Classification of ovoids

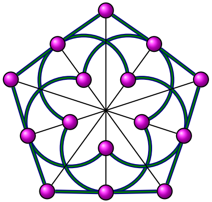
Are the only ovoids the elliptic quadrics and Suzuki-Tits ovoids?

Generalised polygons

Tits (1959): a model for rank 2 simple groups of Lie type.

What are they?

- Rank 2 irreducible spherical buildings.
- Points-line geometries whose incidence graph has $\mathrm{girth} = 2 \cdot \mathrm{diameter}.$
- A generalised 3-gon is a projective plane.



Theorem (Feit-Higman Theorem (1964))

A nontrivial generalised n-gon has $n \in \{3, 4, 6, 8\}$*.*

Known examples (up to duality)

8 One family related to the groups ${}^2F_4(q)$.

6 Two families related to ${}^3D_4(q)$ and $G_2(q)$.

4 Many examples, but not too many! Includes geometries for the groups ${}^2A_3(q)/{}^2D_2(q)$, ${}^2A_4(q)$, $B_2(q)/C_2(q)$

3 Heaps!

Order (s, t)

- every line has $s+1$ points,
- every point lies on $t+1$ lines.

Parameters of generalised polygons

Possible

- 3 $s = t$
- 4 $\sqrt{t} \leqslant s \leqslant t^2$ (Higman Inequality)
- 6 $st \in \Box$ and $\sqrt[3]{t} \leqslant s \leqslant t^3$ (Haemers-Roos Inequality)
- 8 $2st \in \Box$ and $\sqrt{t} \leqslant s \leqslant t^2$ (Higman Inequality)

Known (up to duality)

- 4 $(q, q), (q, q^2), (q^2, q^3), (q-1, q+1)$
- 6 $(q, q), (q, q^2)$
- 8 (q, q^2)

Other big problems

A unital of a projective plane of order q^2 is a set of $q^3 + 1$ points $\mathcal{U}$ such that every line meets $\mathcal{U}$ in 1 or $q + 1$ points.

Buekenhout-Metz Conjecture

Every unital of the Desarguesian projective plane arises from the construction of Buekenhout and Metz.

- Been around since 1979.
- True for $q \in \{4, 9, 16\}$.
- A counter-example has a smallish group.

Other big problems

A *k-blocking set* of a proj. space $\mathbb{P}V$ is a set of points $\mathcal{B}$ such that every k-codimensional space meets $\mathcal{B}$.

Small Blocking Sets Conjecture

A minimal k-blocking set in $\mathbb{PF}_q^n$ of size less than $3(q^k+1)/2$ is a *linear set*.

- Formulated in 2008, but can be traced to earlier origins.
- True for $q = p^3$ (Lavrauw, Storme, Van de Voorde 2011)

Other big problems

A maximal arc of a projective plane is a set of points $\mathcal{A}$ such that every line meets $\mathcal{A}$ in 0 or m points.

Maximal Arcs Conjecture

Maximal arcs exist only in projective planes of even order.

- Been around since 1975 (J. A. Thas).
- True for Desarguesian planes (Ball, Blokhuis, Mazzocca 1997).

Other big problems

A *k-arc* of $\mathbb{PF}_q^n$ is a set of k points, no n lying in a hyperplane.

Example

Normal rational curve (classical):

$$\{(1, t, t^2, t^3, \ldots, t^{n-1})\colon t \in \mathbb{F}_q\} \cup \{(0, 0, \ldots, 0, 1)\}.$$

$(q+1)$-arcs

Does there exist a non-classical $(q+1)$−arc of $\mathbb{PF}_q^n$, $n \leqslant q$, q odd, that is not the *Glynn 10-arc* of $\mathbb{PF}_9^5$?