New Types of Chromatic Factorisation

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Some joint work with Graham Farr

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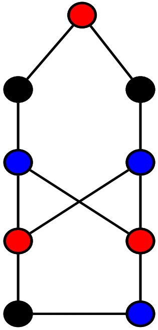
Overview

- Chromatic Polynomial
- Properties
- Chromatic Factorisation
- Certificates

The Chromatic Polynomial

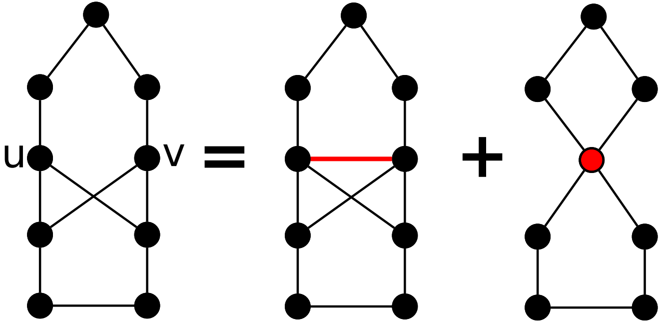
$$P(G;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)(\lambda^4-6\lambda^3+15\lambda^2-19\lambda+11)$$

- Gives the number of proper colourings in at most λ colours
- An algebraic attempt to solve 4CT (Birkhoff, 1912)
- Chromatic number $\chi(G) = 3$



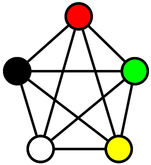
Addition-identification

$$P(G;\lambda) = P(G+uv;\lambda) + P(G/uv;\lambda)$$



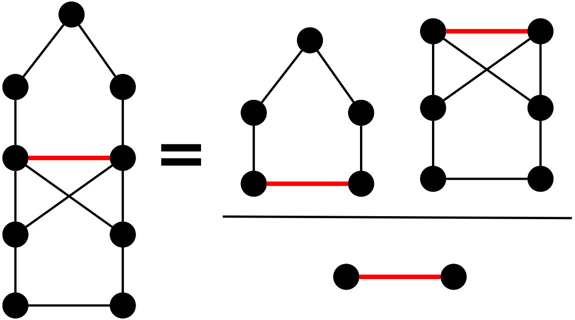
The complete graph of order r:

$$P(K_r) = \lambda(\lambda - 1)\dots(\lambda - r + 1)$$



Clique-separable Graphs

$$P(G;\lambda) = \frac{P(H_1;\lambda) \times P(H_2;\lambda)}{P(K_r;\lambda)}$$



Chromatic Factorisation

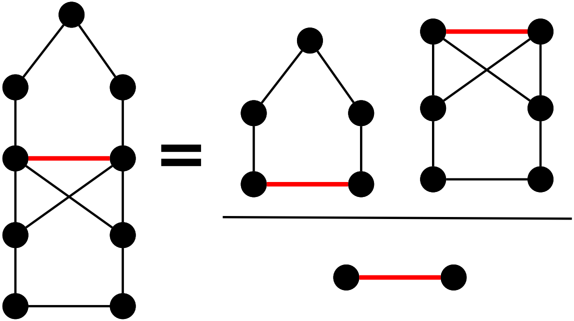
"It seems that there are now two outstanding problems in the theory of chromials [chromatic polynomials] of triangulations: when do chromials factorize, and what is the significance of the Beraha numbers?" (Tutte, 1972)

Chromatic Factorisation

A graph G has a *chromatic factorisation* of order r with *chromatic factors* H_1 and H_2 if

$$P(G;\lambda) = \frac{P(H_1;\lambda) \times P(H_2;\lambda)}{P(K_r;\lambda)}$$

where $r \leq \min\{\chi(H_1), \chi(H_2)\}$.



Chromatic factorisation

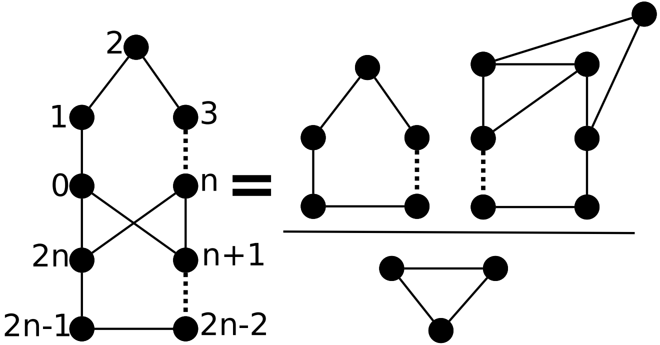
Graphs that have chromatic factorisations include:

1. Clique-separable graphs
2. Graphs chromatically equivalent to clique-separable graphs
3. Others!

n	# Chromatic Polys	# Graphs
8	2	3
9	25	97
10	485	3018
$8 \leq 9 \leq 10$	512	3118

Table: Strongly non-clique separable graphs that have chromatic factorisations

Chromatic Factorisation



Infinite family of strongly non-clique-separable graphs that have a chromatic factorisation with the odd cycle as a factor. (Morgan & Farr, 2009)

Chromatic factorisation

What if we break the rules?

Revisiting the definition ...

A graph G has a *chromatic factorisation* of order r with *chromatic factors* H_1 and H_2 if

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where $r \leq \min\{\chi(H_1), \chi(H_2)\}$.

Number of chromatic . . .

Type 1 Denominator is K_r and $r > \min(\chi H_1, \chi H_2)$,

Type 2 Denominator is a chordal graph (but not complete graph),

Type 3 Denominator is not a chordal graph.

n	Type 1	Type 2	Type 3	Type 1 and Type 2
7	1	1	0	0
8	16	28	0	8
9	203	324	54	122
10	3809	5415	2245	2299
7–10	4029	5768	2299	2429

9667 chromatic polynomials!

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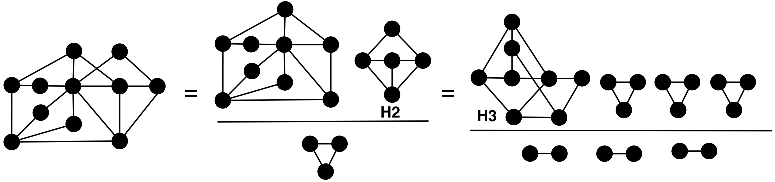
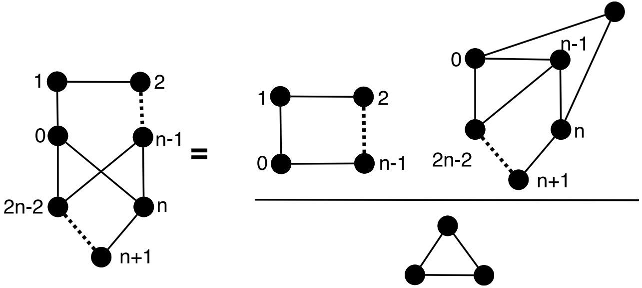
Non-unique Factorisation



- All chromatic factors “irreducible"
- $P(H_2; \lambda)$ divides $P(H_3; \lambda)$
- $P(H_3; \lambda) = P(H_2; \lambda)(\lambda - 2)(\lambda^2 - 3\lambda + 6)$
- H_3 is the graph of smallest order with chromatic polynomial divisible by $(\lambda^2 - 3\lambda + 6)$
- $P(G; \lambda) = P(K_3; \lambda)(\lambda - 2)^4(\lambda^2 - 4\lambda + 5)(\lambda^2 - 3\lambda + 6)$

Denominator Type 1

Denominator is K_r and $r < \min(\chi H_1, \chi H_2)$
(One factor with chromatic number $< r$.)



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$$r = 3 \not< \min(\chi(H_1) = 2, \chi(H_2) = 3).$$

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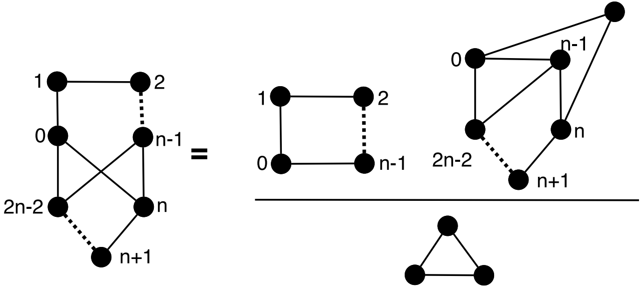

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Certificates

Certificate of factorisation:

- $E_0 = G \to E_1 \to \ldots \to E_k = H_1 H_2 / K_r$
- Graph Operations and algebra
- Proof that G has a chromatic factorisation
- Avoid cost of computing the polynomial

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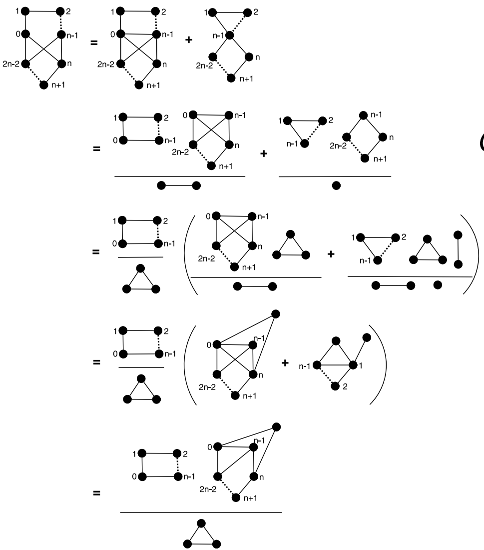

$$G \rightarrow G + uv \; + \; G/uv$$

$$\rightarrow \frac{H_1 H_3}{K_2} + \frac{H_1 H_4}{K_1}$$

$$\rightarrow \frac{H_1}{K_3}\left(\frac{K_3 H_3}{K_2} + \frac{K_3 K_2 H_4}{K_2 K_1}\right)$$

$$\rightarrow \frac{H_1}{K_3}\left(H_5 \; + \; H_6\right)$$

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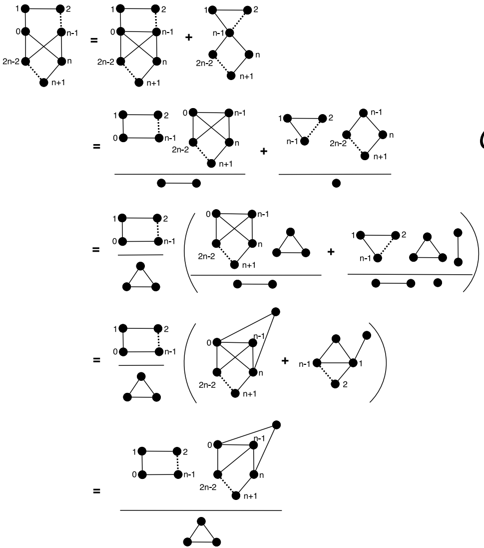
Chromatic Factorisation



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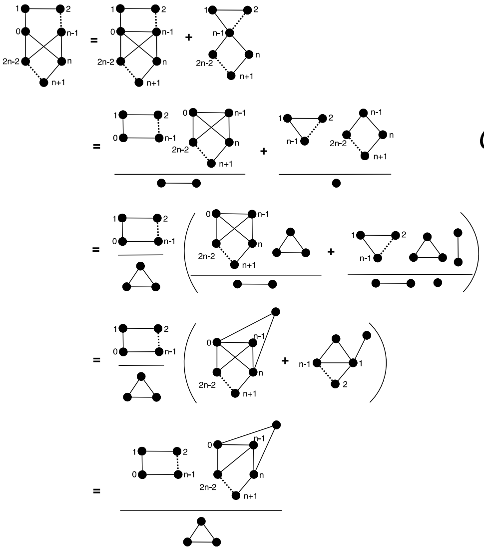

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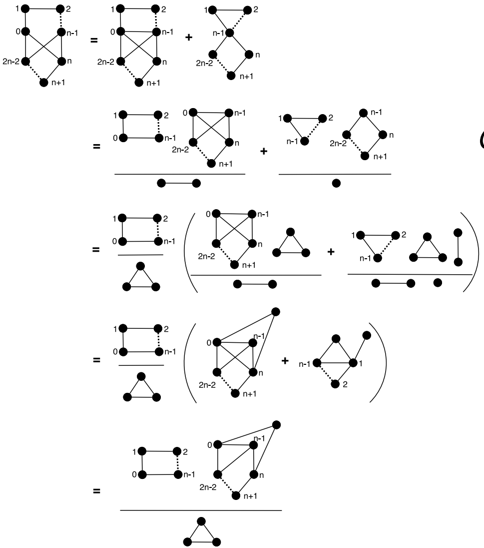

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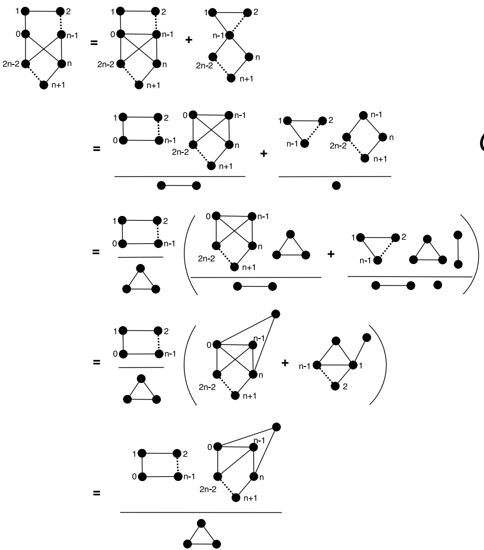

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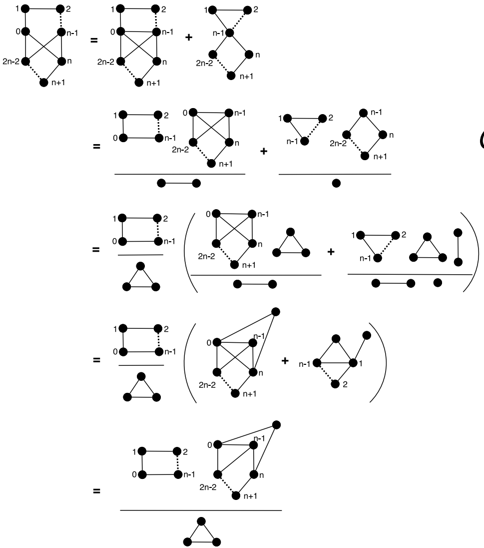

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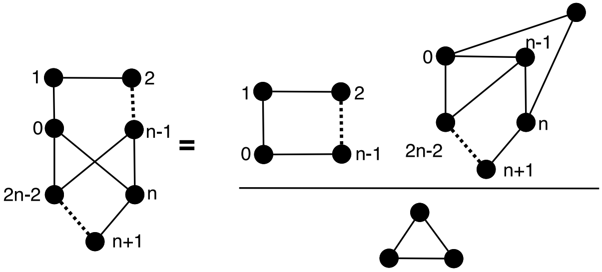
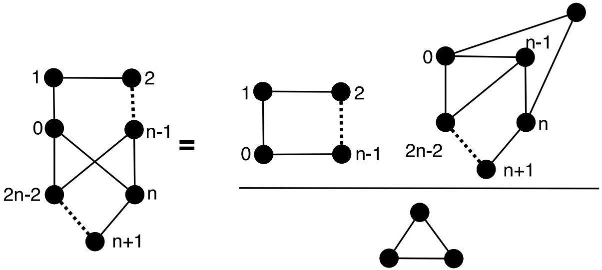


- Infinite family of strongly non-clique-separable graphs of order $2n - 1$ that have chromatic factorisations of order 3.
- For n even, $\chi(C_n) = 2 < 3$.
- For n odd, $\chi(C_n) = 3$.
- Other chromatic factor has chromatic number 3.

Chromatic Factorisation



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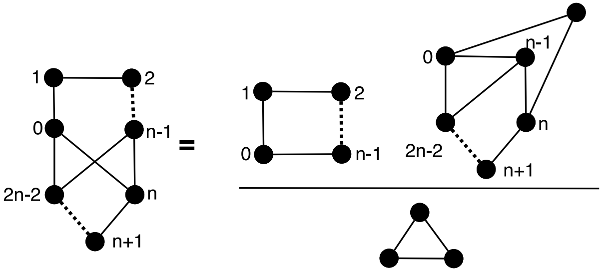

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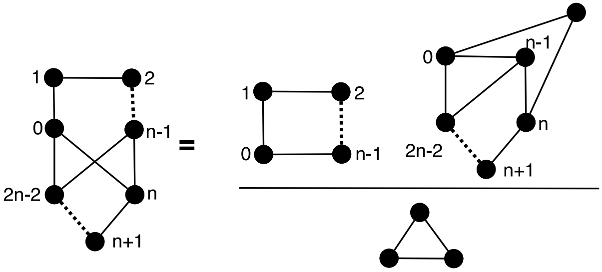

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Chromatic Factors

The cycle $C_{n>3}$ is a chromatic factor of a chromatic factorisation of order 3.

What other graphs can be a chromatic factor of a chromatic factorisation of order 3?

$$P(G;\lambda) = \frac{P(H_1;\lambda) \times P(H_2;\lambda)}{P(K_3;\lambda)}$$

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Chromatic Factors $r = 1$

Any graph H_1 that has at least one block that is not K_1 is a chromatic factor of a chromatic factorisation

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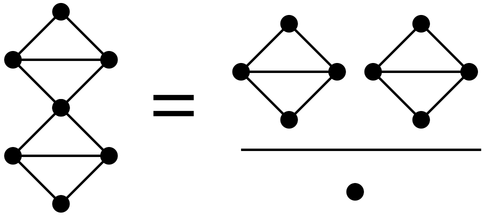
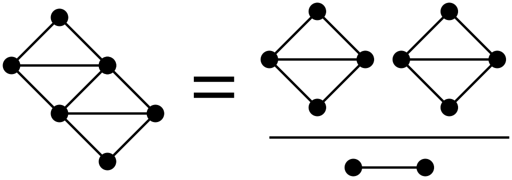


Chromatic Factors $r = 2$

Any graph H_1 that has at least one block that is not K_r, $r \in \{1, 2\}$, is a chromatic factor of a chromatic factorisation

$$P(G; \lambda) = \frac{P(H_1; \lambda)P(H_2; \lambda)}{P(K_2; \lambda)}$$



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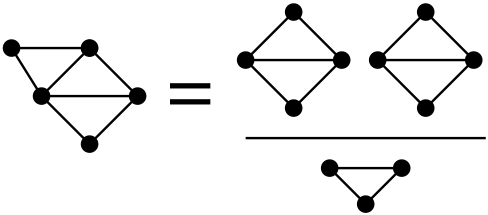

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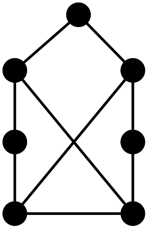
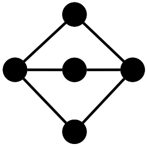


- H_1 may not have a triangle.
  - H_1 does have a cycle.



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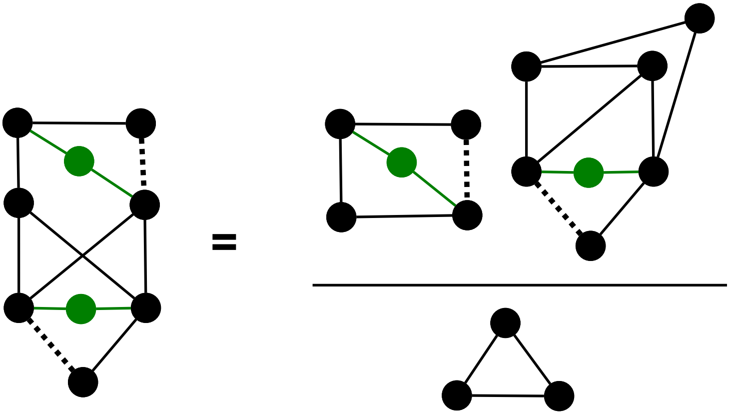

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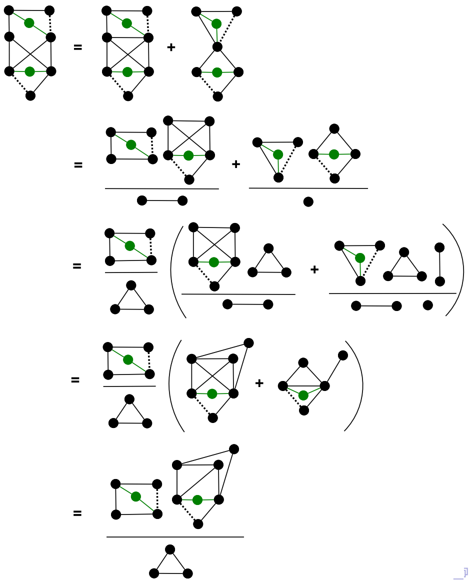

Chromatic Factors

What other graphs can be a chromatic factor of a chromatic factorisation of order 3?

- Any graph that contains a triangle (excluding K_3)
- Cycles of order > 3
- Any H_1 that has a cycle of order > 3.
- What's left???
  - Acyclic graphs
  - K_3
  - All blocks K_r, $r \in \{1, 2, 3\}$.

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Theorem

Any graph H_1 that has at least one block that is not K_r, $r \in \{1, 2, 3\}$, is a chromatic factor of a chromatic factorisation of order 3.

Chromatic Factors

Theorem

Any 2-connected graph H_1 of order ≥ 4 is a chromatic factor of a chromatic factorisation of order 3.

Other denominators

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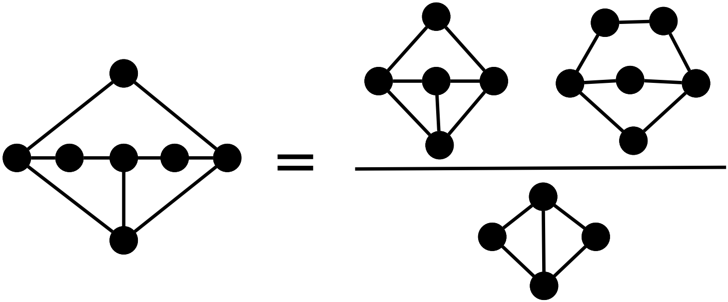
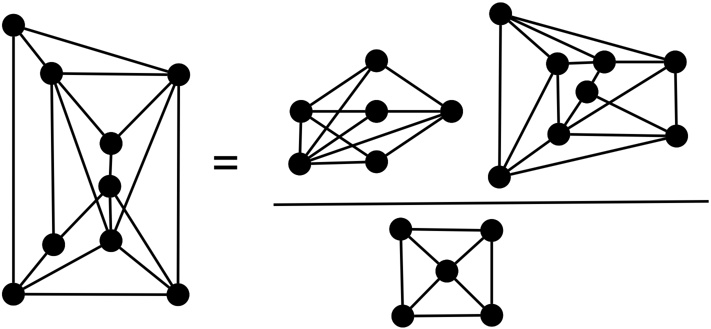
$$P(G;\lambda) = \frac{P(H_1;\lambda) \times P(H_2;\lambda)}{P(K_r;\lambda)P(H_3;\lambda)}$$

where $r \leq \min\{\chi(H_1), \chi(H_2)\}$.

Chordal graphs as a denominator



Non-chordal graphs as a denominator



Certificates

Certificates are a useful tool for proving (or helping to prove) results about algebraic properties of graph polynomials

- Factorisation
- Equivalence
- Divisibility

Tutte's Outstanding Problems

What about the Beraha numbers?

- $B_k = 2 + 2cos\left(\frac{2\pi}{k}\right)$.
- Non-integer Beraha numbers are not chromatic roots ...
- Except for (maybe) B_{10}
- Is $(\lambda^2 - 5\lambda + 5)$ a factor of any chromatic polynomial?
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What about the Beraha numbers?

- $B_k = 2 + 2cos\left(\frac{2\pi}{k}\right)$.
- Non-integer Beraha numbers are not chromatic roots ...
- Except for (maybe) B_{10}
- Is $(\lambda^2 - 5\lambda + 5)$ a factor of any chromatic polynomial?
- What algebraic integers can be chromatic roots?

Other Interests

- $\alpha + n$ conjecture for quartics and beyond
- Other α-style conjectures
- Splitting field equivalence
- Properties of Tutte-equivalent graphs
- What does $T(M; 2, 0)$ count – "Acylic orientations" in binary matroids