Delandtsheer Designs

Mark Ioppolo

University of Western Australia
(Joint with John Bamberg, Alice Devillers and Cheryl Praeger)

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Designs

A $t-(v,k,\lambda)$ block design is an incidence structure

$$\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$$

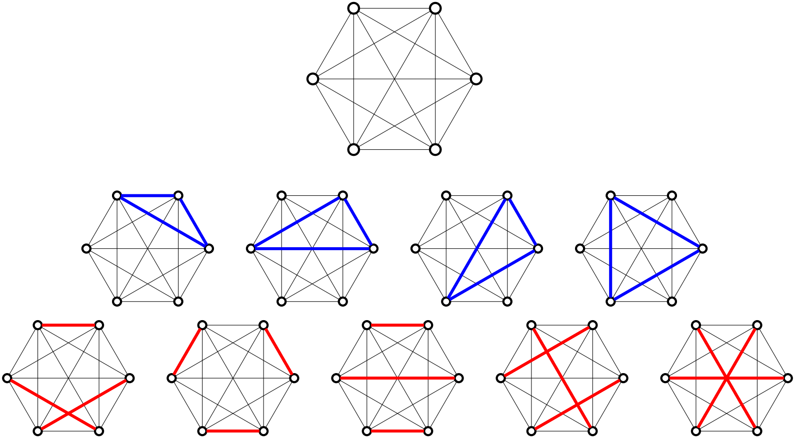
satisfying the following conditions

(i) $|\mathcal{P}| = v$,

(ii) $|B| = k$ for all $B \in \mathcal{B}$,

(iii) For any subset T of t points, there are exactly λ blocks incident with all points in T.

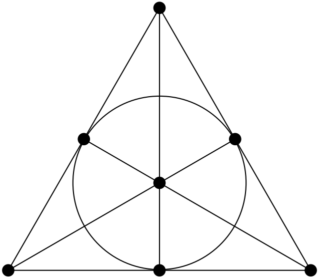
$2-(15,3,1)$ design from K_6



Isomorphic to a design induced by the lines of $PG(3,2)$.

Automorphisms of Designs

An *automorphism* of a design is permutation of the point set which preserves the block set.



The Fano Plane is a $2-(7,3,1)$ design with full automorphism group $\mathrm{PGL}(3,2)$

FAB-Transitive Designs

$$\begin{aligned} \textit{Flag} &: \text{Point block pair } (p, B) \text{ such that } p \in B \\ \textit{Antiflag} &: \text{Point block pair } (p', B) \text{ such that } p' \notin B \end{aligned}$$

Definition

A design is called *FAB-transitive* if there exists $G \leq \mathrm{Aut}(\mathcal{D})$ such that G acts transitively on tuples of the form

$$(p, p', B) \in \mathcal{P} \times \mathcal{P} \times \mathcal{B},$$

where (p, B) is a flag, and (p', B) is an antiflag.

- Necessarily point and block transitive.
- Studied by Delandtsheer (1984) for linear spaces.

Steiner Designs: $\lambda = 1$

Designs with parameters $t-(v,k,1)$ are known as *Steiner t-designs*.

Theorem (1990 - BDDKLS)

The flag-transitive linear spaces are appear on a list or $G \leq \mathrm{A\Gamma L}(1,p^a)$ *and* $|\mathcal{P}| = p^a$.

- Proof appears in two papers; Liebeck 1998 and Saxl 2002.
- Affine case is 'believed to be hopeless'.

Flag-Transitive Steiner t-Designs

Cameron, Praeger (1993)

- There are no flag transitive $t-(v,k,1)$ designs for $t \geq 7$
- Conjectured the non-existence of block transitive Steiner 6-designs

Huber

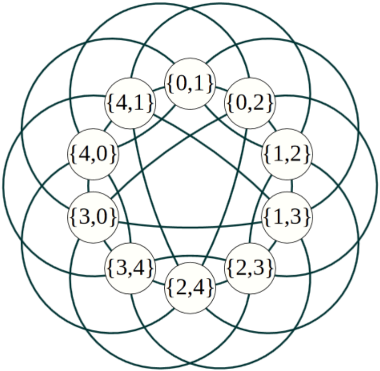
- (2005-2007) Classified flag transitive Steiner t-designs for $3 \leq t \leq 5$
- (2010) Demonstrated non-existence of block transitive Steiner 6-designs, modulo some small open cases

Motivation: Error Correcting Codes

A *code* in a graph Γ is a subset of vertices $\mathcal{C} \subseteq V\Gamma$.

- Johnson graphs $\Gamma = J(\Omega, k)$:

$$V\Gamma: \quad k \text{ element subsets S of } \Omega$$
$$E\Gamma: \quad |S \cap S'| = k - 1.$$



Liebler and Praeger (2014)
Classified most families of neighbour-transitive codes in Johnson graphs.

- Intransitive
- Imprimitive
- Primitive via 2-transitivity (non-symplectic infinite families)

Neunhöffer and Praeger (2013)
Classified primitive type Neighbour-transitive codes which do not lie in an infinite family.

Symplectic Case

Problem (Liebler, Praeger)

Classify the neighbour-transitive codes in Johnson graphs satisfying $\mathrm{Aut}(\Gamma) = \mathrm{Sp}(2n, 2)$ *and* $|\Omega| = 2^{n-1}(2^n + \epsilon)$.

- Correspond to 2-designs with $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{Sp}(2n, 2)$ whose points are *quadratic forms* on $\mathbb{F}_2^{2n}$.

A Baby Lemma

Lemma

A 2-design $\mathcal{D}$ is FAB-transitive if and only if there exists $G \leq \mathrm{Aut}(\mathcal{D})$ such that

(i) *G is transitive on blocks, and*

(ii) *For every block B, G_B acts transitively on $B \times \overline{B}$.*

- Block stabilisers G_B lie in maximal subgroups of $\mathrm{Sp}(2n, 2)$.
- Maximal subgroups of classical groups are well understood; eight 'geometric classes' and the almost-simple class.

Geometric Case

Theorem (Bamberg, Devillers, MI, Praeger)

If $\mathcal{D}$ is a FAB-transitive 2-design such that

(i) $\mathcal{D}$ *has* $2^{n-1}(2^n+\epsilon)$ *points,*

(ii) $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{Sp}(2n,2)$*, and*

(iii) G_B *is a* **geometric type** *subgroup,*

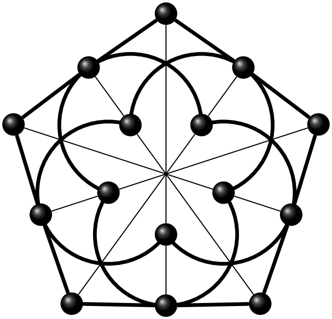
then G_B *fixes a subspace of* $\mathbb{F}_2^{2n}$ *and* $\mathcal{D}$ *is explicitly known.*

Get several infinite families!

Example in $W(3,2)$

The ‘Doily’ is a projective representation of 4-dimensional symplectic space

$$f(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$
$$Q(\mathbf{x}+\mathbf{y}) = Q(\mathbf{x}) + Q(\mathbf{y}) + f(\mathbf{x},\mathbf{y})$$



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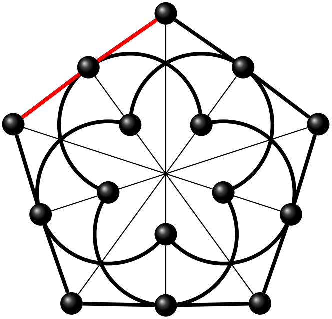

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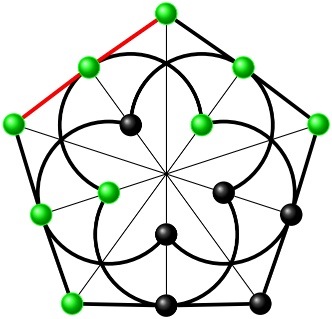
$$f(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$
$$Q(\mathbf{x}) = x_1x_2 + x_3x_4$$



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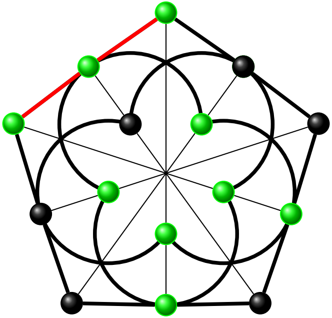
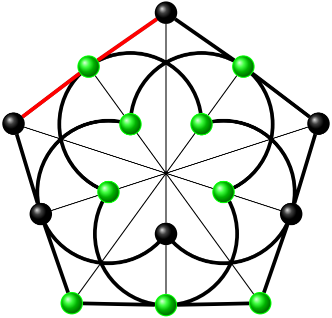
$$f(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$
$$Q(\mathbf{x}) = x_1x_2 + x_3x_4 + x_2^2 + x_4^2$$



Example in $W(3,2)$

The 'Doily' is a projective representation of 4-dimensional symplectic space

$$B(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$
$$Q(\mathbf{x}) = x_1x_2 + x_3x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$



A $\mathcal{C}_9$ subgroup M in $\mathrm{Sp}(2n, 2)$:

(i) Fixes no subspaces of $\mathbb{F}_2^{2n}$, and

(ii) $T \leq M \leq \mathrm{Aut}(T)$ for a simple group T.

Theorem (Liebeck - 1985)

A maximal $\mathcal{C}_9$ subgroup M of a classical group $C(m, q)$ is either isomorphic to a fully deleted permutation module for S_n or $|M| \leq q^{3m}$.

Permutation Modules for S_{2n+2}

S_{2n+2} acts on $\mathbb{F}_2^{2n}$. If n is even and $n \geq 3$ we have

$$\mathrm{O}^{\epsilon}(2n, 2) < S_{2n+2} < \mathrm{Sp}(2n, 2).$$

Theorem (Bamberg, Devillers, MI, Praeger)

If $\mathcal{D}$ is a FAB-transitive 2-design and $G_B \leq S_{2n+2}$ fixes no subspaces $\mathbb{F}_2^{2n}$ then $(n, \epsilon) = (4, +)$ and $\mathcal{D}$ is a known design with 13056 blocks and parameters

$$2-(136, 10, 64), \textit{ or}$$
$$2-(136, 126, 11200).$$

Thanks!

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