Relations between partitions: some problems

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University of St Andrews / QMUL (emerita)

University of Western Australia, September 2016
An example of three uniform partitions of the same set

The underlying set has size 36 (vegetable patches).

- The partition $D$ into districts has 4 parts of size 9.
- The partition $G$ into gardens has 12 parts of size 3.
- The partition $L$ into letters (lettuce varieties) has 9 parts of size 4.

Three binary relations:

- $G \preceq D$, $G$ is a refinement of $D$;
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Some definitions for a uniform partition of a finite set

Ω is the underlying set, of size M.

$V_0 = \text{subspace of } \mathbb{R}^\Omega \text{ consisting of constant vectors}.$

For a given uniform partition $F$:

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- \( P_F = \frac{1}{k_F} X_F X_F^\top \) = matrix of orthogonal projection onto \( V_F \), which averages each vector over each part of \( F \).
Some definitions for two uniform partitions of the same set

\[ N_{FG} = X_F^T X_G \]
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**Orthogonality** $F \perp G$ means the following equivalent things:

- $N_{FG} = \text{constant} \times J$;
- $P_F P_G = P_G P_F = P_0 = \text{projector onto } V_0$;
- $(V_F \cap V_0^\perp) \perp (V_G \cap V_0^\perp)$;
- $X_F^T (I - P_0) X_G = 0$;
- $N_{F0} N_{0G} = k_0 N_{FG} = MN_{FG}$. 

**Balance** If no entry in $N_{FG}$ is bigger than 1 then $F \triangle G$ means the following equivalent things:

- $N_{FG} N_{GF}$ is completely symmetric (a linear combination of $I$ and $J$) but not scalar;
- $X_F^T (I - P_G) X_F$ is completely symmetric but not zero.

(We usually exclude orthogonality.)
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So you can have fun making sets of partitions on the same set such that each pair is related by refinement or orthogonality or balance.
Ternary relations?

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**Adjusted orthogonality** Partitions $F$ and $G$ have adjusted orthogonality with respect to partition $L$ if

$$X_F^\top (I - P_L)X_G = 0;$$

equivalently,

$$N_{FL}N_{LG} = k_L N_{FG}.$$
If $\mathcal{L}$ is a set of partitions of $\Omega$, put

$$P_\mathcal{L} = \text{matrix of orthogonal projection onto } \sum_{L \in \mathcal{L}} V_L.$$
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$$X_F^\top (I - P_{\mathcal{L}}) X_G = 0.$$
An $r \times c$ triple array is an $r \times c$ rectangle, each cell containing one of $r + c - 1$ letters, such that

- rows $R$ are strictly orthogonal to columns $C$, with all intersections of size 1;
- rows are balanced with respect to letters ($L$) (every pair of rows has the same number of letters in common);
- columns are balanced with respect to letters;
- rows and columns have adjusted orthogonality with respect to $L$ (the set of letters in each row has constant size of intersection with the set of letters in each column).
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Put the letters in cells and obtain these subsets in rows and columns.
How did I make it? Start with a SBIBD

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Row name is not in

Put the letters in cells and obtain these subsets in rows and columns
Problem: can you do it?

Given a subset of letters allowed for each cell, is it possible to choose an array of distinct representatives, one per cell, so that no letter is repeated in a row or column?
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Your task: Proof or counter-example.
Balance among three or more uniform partitions

\[ P_F = \text{matrix of orthogonal projection onto } V_F \]
\[ P_0 = \text{matrix of orthogonal projection onto } V_0 \]

Put \( Q_F = P_F - P_0 \).
Balance among three or more uniform partitions

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\( F \) is balanced with respect to \( G \) means
(in addition to banning orthogonality)
\( N_{FG}N_{GF} \) is completely symmetric but not scalar; equivalently
\( X_F^T(I - P_G)X_F \) is completely symmetric but not zero; equivalently, \( Q_FQ_GQ_F \) is a non-zero scalar multiple of \( Q_F \).
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If \( G \) is a set of partitions of \( \Omega \),

\[ P_G = \text{matrix of orthogonal projection onto } \sum_{G \in G} V_G \].
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If \( \mathcal{G} \) is a set of partitions of \( \Omega, \)
\[ P_{\mathcal{G}} = \text{matrix of orthogonal projection onto } \sum_{G \in \mathcal{G}} V_G. \]

\( F \) is \textbf{balanced with respect to} \( \mathcal{G} \) if
\[ X_F^\top (I - P_{\mathcal{G}})X_F \] is completely symmetric but not zero.
Suppose that partitions $F$, $G$ and $H$ each have $n$ parts of size $k$, and that each pair are balanced (both ways).
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Then $F$ is balanced with respect to $\{G,H\}$ if and only if

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF}$$

is completely symmetric.

Equivalently,

$$Q_F(Q_G Q_H + Q_H Q_G) Q_F$$

is a non-zero multiple of $Q_F$. 
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Equivalently,

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The above is implied by this stronger condition:

$$N_{FG}N_{GH} \text{ is a linear combination of } N_{FH} \text{ and } J.$$
A set $\mathcal{L}$ of uniform partitions of $\Omega$, all with $n$ parts, has **universal balance** if whenever $F \in \mathcal{L}$ and $G \subseteq \mathcal{L} \setminus \{F\}$ then $F$ is balanced with respect to $G$. 

Equivalently (I hope), whenever $F$ and $G$ are as above, then 

$$Q_F \left( \sum_{\sigma \in \text{Sym}(r)} \sigma Q_G(1) \sigma(2) \cdots \sigma(r) \right) Q_F$$

is a non-zero multiple of $Q_F$, where $r = |G|$. 

Equivalently, 

$$\sum_{\sigma} N_{FG} \sigma(1) N_{G} \sigma(1) \cdots N_{G} \sigma(r) F$$

is . . .
My attempt at a general definition

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Equivalently (I hope), whenever $F$ and $\mathcal{G}$ are as above, then

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Equivalently, $\sum_{\sigma} N_{FG_{\sigma(1)}} N_{G_{\sigma(1)}G_{\sigma(2)}} \cdots N_{G_{\sigma(r)}F}$ is ...
Known families, for $n$ parts of size $k$

\[ N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \] is completely symmetric, or its generalization.
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$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF}$$ is completely symmetric, or its generalization.

- $k = n - 1$: remove a common transversal from a set of mutually orthogonal $n \times n$ Latin squares, so that every $N$ is $J - I$. 

$n \equiv 3 \pmod{4}$ and $k = (n + 1)/2$ or $k = (n - 1)/2$: if there is a doubly-regular tournament of size $n$, its adjacency matrix $A$ satisfies $I + A + A^\top = J$ and $A^2 \in \langle I, A, J \rangle$, then ensure that each $N$ is either $I + A$ or $I + A^\top$ (or $A$ or $A^\top$).

$n = 2^2 m$ and $k = 2^2 m - 1 + 2^m - 1$ or $k = 2^2 m - 1 - 2^m - 1$: Cameron has constructions from quadratic forms, and the strong form of the condition is satisfied. (For $n = 16$ and $k = 6$ this involves compatible Clebsch graphs which form an amorphic association scheme.)
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- \( n = 2^m \) and \( k = 2^{m-1} + 2^{m-1} \) or \( k = 2^{m-1} - 2^{m-1} \): Cameron has constructions from quadratic forms, and the strong form of the condition is satisfied. (For \( n = 16 \) and \( k = 6 \) this involves compatible Clebsch graphs which form an amorphic association scheme.)
Problem: is this all?

Your task

- Find all possible sets of three or more incidence matrices $N_{FG}$ satisfying the conditions.
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- For each such set, realise them as incidence matrices of a set of partitions.

Or three or more?
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- For each such realisation, find another partition with $k$ parts of size $n$ that is orthogonal to all the rest (surprisingly, this often makes the previous part easier).
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