Relations between partitions: some problems

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An example of three uniform partitions of the same set

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C	F	I

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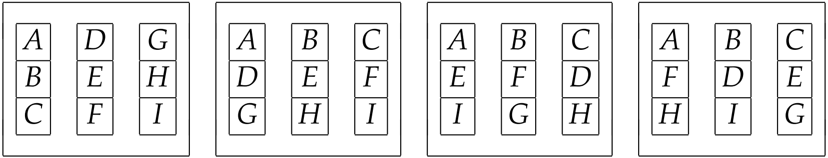

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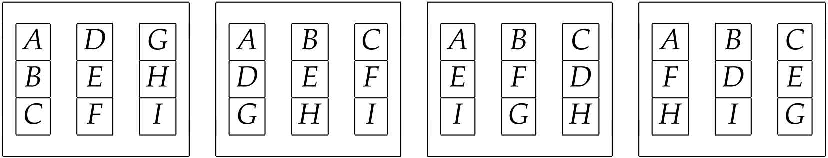

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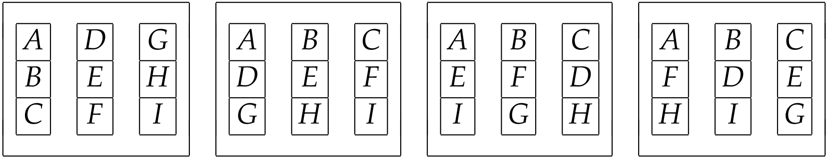

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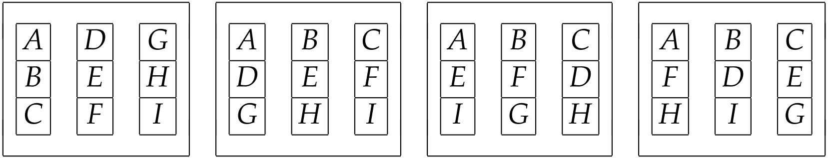

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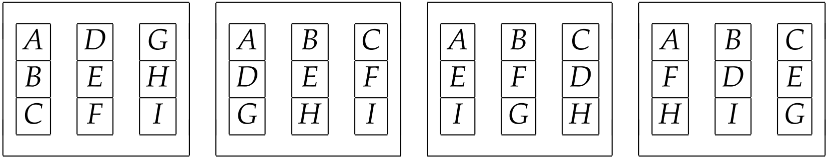

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Some definitions for a uniform partition of a finite set

Ω is the underlying set, of size M.
$V_0 =$ subspace of $\mathbb{R}^\Omega$ consisting of constant vectors.

For a given uniform partition F:

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- X_F is the $M \times n_F$ incidence matrix of elements of Ω in parts of F;
- $P_F = \frac{1}{k_F} X_F X_F^\top =$ matrix of orthogonal projection onto V_F, which averages each vector over each part of F.

Some definitions for two uniform partitions of the same set

$$
\begin{aligned}
N_{FG} &= X_F^\top X_G \\
&= \text{incidence matrix between parts of } F \text{ and parts of } G.
\end{aligned}
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Orthogonality $F \perp\!\!\!\perp G$ means the following equivalent things:

- $N_{FG} = \text{constant} \times J$;
- $P_F P_G = P_G P_F = P_0 =$ projector onto V_0;
- $\left(V_F \cap V_0^\perp\right) \perp \left(V_G \cap V_0^\perp\right)$;
- $X_F^\top (I - P_0) X_G = 0$;
- $N_{F0} N_{0G} = k_0 N_{FG} = M N_{FG}$.

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Balance If no entry in N_{FG} is bigger than 1
then $F \rhd G$ means the following equivalent things:

- $N_{FG} N_{GF}$ is completely symmetric
  (a linear combination of I and J) but not scalar;
- $X_F^\top (I - P_G) X_F$ is completely symmetric but not zero.

(We usually exclude orthogonality.)

Ternary relations?

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Adjusted orthogonality Partitions F and G have adjusted orthogonality with respect to partition L if

$$X_F^\top (I - P_L) X_G = 0;$$

equivalently,

$$N_{FL} N_{LG} = k_L N_{FG}.$$

Or higher order?

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Triple arrays

	0	2	6	7	8	X
1	B	A	E	D	J	F
4	G	H	B	I	D	E
9	J	I	A	B	C	G
5	F	J	H	C	E	I
3	H	D	C	F	G	A

An $r \times c$ **triple array** is an $r \times c$ rectangle,
each cell containing one of $r + c - 1$ letters, such that

- rows R are strictly orthogonal to columns C, with all intersections of size 1;
- rows are balanced with respect to letters (L) (every pair of rows has the same number of letters in common);
- columns are balanced with respect to letters;
- rows and columns have adjusted orthogonality with respect to L (the set of letters in each row has constant size of intersection with the set of letters in each column).

How did I make it? Start with a SBIBD

	A	B	C	D	E	F	G	H	I	J
1	2	3	4	5	6	7	8	9	X	0
4	5	6	7	8	9	X	0	1	2	3
9	X	0	1	2	3	4	5	6	7	8
5	6	7	8	9	X	0	1	2	3	4
3	4	5	6	7	8	9	X	0	1	2

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	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
1	2	3	4	5	6	7	8	9	*X*	0
4	5	6	7	8	9	*X*	0	1	2	3
9	*X*	0	1	2	3	4	5	6	7	8
5	6	7	8	9	*X*	0	1	2	3	4
3	4	5	6	7	8	9	*X*	0	1	2

	0	2	6	7	8	*X*
1						
4						
9						
5						
3						
	B	*A*	*A*	*B*	*C*	*A*
	F	*D*	*B*	*C*	*D*	*E*
column name is in	*G*	*H*	*C*	*D*	*E*	*F*
	H	*I*	*E*	*F*	*G*	*G*
	J	*J*	*H*	*I*	*J*	*I*

A	*B*	*D*	*E*	*F*	*J*	
B	*D*	*E*	*G*	*H*	*I*	
A	*B*	*C*	*G*	*I*	*J*	row name is not in
C	*E*	*F*	*H*	*I*	*J*	
A	*C*	*D*	*F*	*F*	*H*	

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9	X	0	1	2	3	4	5	6	7	8
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	0	2	6	7	8	X
1						
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column name is in	B F G H J	A D H I J	A B C E H	B C D F I	C D E G J	A E F G I

A	B	D	E	F	J	row name is not in
B	D	E	G	H	I	
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Put the letters in cells and obtain these subsets in rows and columns

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	0	2	6	7	8	*X*
1				*BDF*		
4						
9						
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3						
	B	*A*	*A*	*B*	*C*	*A*
	F	*D*	*B*	*C*	*D*	*E*
column name is in	*G*	*H*	*C*	*D*	*E*	*F*
	H	*I*	*E*	*F*	*G*	*G*
	J	*J*	*H*	*I*	*J*	*I*

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Put the letters in cells and obtain these subsets in rows and columns

Problem: can you do it?

Given a subset of letters allowed for each cell,
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Your task: Proof or counter-example.

Balance among three or more uniform partitions

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(in addition to banning orthogonality)
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Suppose that partitions F, G and H each have n parts of size k, and that each pair are balanced (both ways).

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Then F is balanced with respect to $\{G, H\}$ if and only if

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric.}$$

Equivalently,

$$Q_F(Q_GQ_H + Q_HQ_G)Q_F \text{ is a non-zero multiple of } Q_F.$$

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The above is implied by this stronger condition:

$$N_{FG}N_{GH} \text{ is a linear combination of } N_{FH} \text{ and } J.$$

My attempt at a general definition

A set $\mathcal{L}$ of uniform partitions of Ω, all with n parts,
has **universal balance** if
whenever $F \in \mathcal{L}$ and $\mathcal{G} \subseteq \mathcal{L} \setminus \{F\}$
then F is balanced with respect to $\mathcal{G}$.

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Equivalently (I hope), whenever F and $\mathcal{G}$ are as above, then

$$Q_F \left(\sum_{\sigma \in \mathrm{Sym}((r)} Q_{G_{\sigma(1)}} Q_{G_{\sigma(2)}} \cdots Q_{G_{\sigma(r)}} \right) Q_F$$

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Equivalently, $\sum_{\sigma} N_{FG_{\sigma(1)}} N_{G_{\sigma(1)}G_{\sigma(2)}} \ldots N_{G_{\sigma(r)}F}$ is …

Known families, for n parts of size k

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- $n = 2^{2m}$ and $k = 2^{2m-1} + 2^{m-1}$ or $k = 2^{2m-1} - 2^{m-1}$: Cameron has constructions from quadratic forms, and the strong form of the condition is satisfied. (For $n = 16$ and $k = 6$ this involves compatible Clebsch graphs which form an amorphic association scheme.)

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