Symmetries of generalised polygons

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Generalised n-gon: point–line incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ with

(i) every two elements of $\mathcal{P} \cup \mathcal{L}$ in some ordinary n-gon;

(ii) no ordinary k-gons for $k < n$.

If Γ is thick (at least three points on each line, and vice versa) then

- it has an order (s, t): each line (point) has $s + 1$ points ($t + 1$ lines);
- $n \in \{2, 3, 4, 6, 8\}$ (Feit–Higman, 1964).

Known examples:

n	classical, with group	non-classical
4	$\mathrm{PSp}_4(q)$, $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$	lots, constructed ‘synthetically’
6	${}^3\mathrm{D}_4(q)$, $\mathrm{G}_2(q)$	none known
8	${}^2\mathrm{F}_4(q)$	none known

A complete classification seems too hard, so we ask for symmetry.

The Moufang condition characterises the classical examples. (Tits–Weiss, 2002; also follows from Fong–Seitz, 1973.)

Point-distance-transitivity characterises the classical examples, with one exception: $\mathrm{GQ}(3,5)$. (Buekenhout–Van Maldeghem, 1994.)

If $n = 4$ then antiflag-transitivity characterises the classical examples plus $\mathrm{GQ}(3,5)$ and its dual. (Bamberg–Li–Swartz, *Trans AMS*.)

The classical examples are point- and line-primitive and flag-transitive. There are also point-primitive, flag-transitive but line-imprimitive GQs.

We seek a classification subject to a subset of these conditions.

$\Gamma = (\mathcal{P}, \mathcal{L})$ a (finite, thick) generalised n-gon, $n \in \{4, 6, 8\}$; $G \leqslant \mathrm{Aut}(\Gamma)$.

If G is point-primitive, line-primitive and flag-transitive, then G is an almost simple group of Lie type.

- $n = 6, 8$: Schneider–Van Maldeghem, 2008.
- $n = 4$: Bamberg–Giudici–Morris–Royle–Spiga, 2012.

If $n = 4$ and G is point-primitive and line-transitive, then

- G has a unique minimal normal subgroup M, so is not primitive of type HS or HC (Bamberg–P–Praeger, *J Group Theory*);
- if M is abelian then Γ is the unique GQ$(3, 5)$ or the GQ$(15, 17)$ constructed from the 'Lunelli–Sce hyperoval' (BPP–Glasby, 2014).

We weaken the assumptions further, to point-primitivity alone.

Compare with Kantor (1987): if $n = 3$ then Γ (a projective plane) is Desarguesian, or $|\mathcal{P}| = p$ and G contains a point-regular C_p.

If $n = 6, 8$ then G is

- almost simple of Lie type (BGPP–Schneider, 2014+),
- not of type ${}^2B_2$ of ${}^2G_2$ (Morgan–P, 2016),
- of type ${}^2F_4$ if and only if Γ is the classical generalised octagon or its dual (Morgan–P, 2016).

Proofs depend on various geometric/combinatorial properties of generalised $6, 8$-gons that do not hold for generalised 4-gons:

- non-existence of ordinary quadrangles,
- certain number-theoretic properties related to the order of Γ.

Now take $n = 4$ with G primitive on $\mathcal{P}$ (only).

For G affine, $\Gamma = \mathcal{T}_2^*(\mathcal{O})$, $\mathcal{O} \subset \mathrm{PG}(2, 2^{3m-1})$ a hyperoval with irreducible stabiliser (De Winter–Thas, 2006). A full classification seems hard.

Else we bound the number of points fixed by an automorphism:

Theorem (BPP, 2016+): if $\Gamma \neq \mathrm{GQ}(2,4)$ then $\mathrm{fix}_{\mathcal{P}}(g) < |\mathcal{P}|^{4/5}$, $\forall g \in G$.

This may be compared with a recent result of Babai (2015) about (more or less) arbitrary strongly regular graphs on ℓ vertices, which says that an automorphism can fix at most $7\ell/8$ vertices. Closer inspection of Babai's paper yields an $O(\ell^{7/8})$ bound for GQs, but our sharpening of $7/8 \to 4/5$ is quite useful in this setting.

If G has type HS then $\mathcal{P} = T$ for some simple group T, and G has a subgroup $N = T \times T$ acting by $y^{(x,z)} = z^{-1}yx$. In particular, $(x, x) \in N$ fixes $|C_T(x)|$ points, so the Theorem implies $|C_T(x)| < |T|^{4/5}$, $\forall x \in T$.

Not many families of simple groups satisfy this bound, and we can rule out some that do by other arguments. The other 'diagonal' primitive actions are similar. Here M denotes a minimal normal subgroup of G:

type	M	$r \leqslant$	possibilities for T
HS	T	–	A_n^ε, D_n^ε, $n \leqslant 6$; B_n, C_n, $n \leqslant 3$; E_7, E_6^ε, F_4, G_2, ${}^3D_4$
HC	T^r	2	A_n^ε, $n \leqslant 2$; ${}^2G_2$, ${}^2B_2$
SD	T^k	–	Lie rank $\leqslant 8$; any sporadic; Alt_n, $n \leqslant 18$
CD	$(T^k)^r$	3	$r = 2$: Lie rank $\leqslant 3$; all but 5 sporadics; Alt_7, Alt_9
			$r = 3$: $A_1(q)$, ${}^2B_2(q)$, J_1

Roughly, HS/HC is easier than SD/CD because (i) no dependence on 'k', (ii) extra structure (two regular subgroups).

If G is AS with socle T, then our Theorem rules out a primitive action with point stabiliser H if $f(G) := \mathrm{fix}_{[G:H]}(G) \geqslant |G:H|^{4/5}$.

A classification of all such actions seems a little way off:

- Liebeck–Shalev show that generically $f(G) \geqslant |G:H|^{1/6}$;
- a student of Tim Burness (Elisa Covato) is sharpening $1/6 \to 4/9$.

PA type is easier: $M = T^r$ with product action on $\mathcal{P} = [G:H]^r$, so an element $(*, 1, \ldots, 1)$ fixes $|G:H|^{r-1}$ points and the Theorem implies $r - 1 < 4r/5$, i.e. $r \leqslant 4$. Covato's thesis should give $r \leqslant 2$ generically.

TW type seems very hard: there is a regular normal subgroup T^k, but not necessarily much else. Regular groups on GQs are hard (Ghinelli, 1992: $s = t$; Yoshiara, 2007: $s = t^2$). Fixity of TW groups seems hard to bound (Liebeck–Shalev prove $|T^k|^{1/3}$, but this is too small for us).